# Fuchsian Singularities of Linear Ordinary Differential Equations in Banach Algebras

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## Preface

*My doctoral thesis of 1980 [Al] left unanswered the question, whether the factors of automorphy of a Fuchsian differential equation in an arbitrary unital Banach algebra belong to the range of the exponential function - which is clearly true in matrix algebras and provides a useful tool there .*

*In this paper I will show that the answer in general is* **no** *and prove some sufficient or necessary conditions for the answer to be* **yes**.

*While in the scarce literature on this subject one usually finds attempts to generalize methods known from the classical theory by imposing rather crude assumptions on the abstract cases which are automatically satisfied in the matrix case, I found it more suitable to start only with the most basic facts about Banach algebras, bounded linear operators and holomorphic vector-valued functions.*

*The resulting theory is still far from being complete but already sheds some new light even on the matrix case : e.g. the classical structure theorem on fundamental solutions of Gantmacher which seemed to be a prominent application of matrix methods essentially holds for the broader class of Banach algebras where each element has its spectrum totally disconnected.*

*Some of the considered examples indicate that the class of complex, unital, symmetric commutative semi-simple Banach algebras may play a dominant rôle in the applications.*

*Not surprisingly this paper raises more open questions than it started with ; a simple looking problem formulated by E. Hille in 1969 which triggered much of the present work still has to be solved.*

*The results of this paper were achieved during the years 1983 through 1996 beside my usualwork in the lowlands of industry. A preliminary subset of them was presented to the* Oberseminar *of Burmann and Holdgrün at Göttingen in July 1993. The present English version corrects some minor errors and leaves out all details not explicitely necessary for the main arguments.*

*Some chapters still require editorial work as some tentative arguments are not worked out completely.*

*Wuppertal, May 1996* *G. Albrecht*

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## Contents:

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## Symbols

| | |
|---|---|
| $\emptyset$ | void set . |
| $[a,b]$ | compact interval in $\mathbb{R}$ . |
| $\alpha_q$ | inner automorphism to $q \in \mathcal{B}^*$ : $\alpha_q(x) = qxq^{-1}$ for all $x \in \mathcal{B}$ , $\alpha_q \in BL(\mathcal{B})^*$ . |
| $\mathcal{B}$ | complex Banach algebra with norm $\|.\|$ and unit **e** . |
| $\mathcal{B}^*$ | (topological) group of regular (= invertible) elements in $\mathcal{B}$ . |
| $\mathcal{B}^*_0$ | connected component of the unit in $\mathcal{B}^*$ . |
| BL(X) | Banach algebra of bounded linear operators on a Banach space X . |
| $\mathbb{C}$ | field of complex numbers . |
| $C_a$ | commutator to $a \in \mathcal{B}$ : $C_a(x) = ax-xa$ for all $x \in \mathcal{B}$ , $C_a \in BL(\mathcal{B})$. |
| $C(\Omega)$ | Banach algebra of continuous, complex-valued functions on a compact Hausdorff space $\Omega$ . |
| $C_{\mathbb{R}}(\Omega)$ | real subalgebra of real-valued functions in $C(\Omega)$. |
| $C[a,b]$ | $= C([a,b])$ . |
| $\exp(\mathcal{B})$ | range of the exponential function in a Banach algebra $\mathcal{B}$. |
| $\mathrm{Im}\,\lambda$ | imaginary part of $\lambda \in \mathbb{C}$ . |
| Im T | image of a map T . |
| $\int_{[a,b]} f(s)ds$ | definite (Riemann-)Integral of a continuous function on [a,b] |
| Ker T | kernel of a homomorphism of algebras, groups or the like. |
| $L_a$ | left multiplication with $a \in \mathcal{B}$ : $L_a(x) = ax$ for all $x \in \mathcal{B}$ , $L_a \in BL(\mathcal{B})$. |
| $L^p[a,b]$ | Banach space of the (classes of complex-valued) p-integrable functions in the sense of Lebesgue on [a,b]. |
| $L^\infty[a,b]$ | Banach space of (classes of) essentially bounded functions on [a,b]. |
| $M^c$ | commutant (=centralizer) of a subset $M \subset \mathcal{B}$ , $M^c = \{ x \in \mathcal{B} : xm = m x \text{ for all } m \in M \}$ |
| $M_n(\mathbb{C})$ | Banach algebra of the complex nxn-matrices with a suitable norm. |
| $\mathbb{N}$ | set of natural numbers. |
| $\mathcal{O}^{\mathcal{B}}$ | Sheaf of germs of holomorphic $\mathcal{B}$-valued functions on a Riemann surface |
| $\mathbb{P}_1$ | Riemann sphere . |
| PSp(T) | point spectrum of $T \in BL(X)$ : $PSp(T) = \{ \lambda \in \mathbb{C}: \exists\, x \in X \text{ with } x \neq 0 \text{ and } T(x) = \lambda x \}$. |
| $\mathbb{Q}$ | the field of rational numbers. |
| $\mathbb{R}$ | the field of real numbers. |
| $\rho(x)$ | spectral radius of $x \in \mathcal{B}$ : $\rho(x) = \max \{ \lvert\lambda\rvert : \lambda \in Sp(x) \}$ . |
| R(x) | resolvent set of $x \in \mathcal{B}$ : $R(x) = \mathbb{C} \setminus Sp(x)$ . |
| $R_0(x)$ | unbounded component of R(x) . |
| $\mathrm{rad}(\mathcal{B})$ | (Jacobson) radical of the Banach algebra $\mathcal{B}$ . |
| $R_a$ | right multiplication by $a \in \mathcal{B}$ : $R_a(x) = xa$ for all $x \in \mathcal{B}$ , $R_a \in BL(\mathcal{B})$. |
| $\mathrm{Rea}\,\lambda$ | real part of $\lambda \in \mathbb{C}$ . |
| $S^1$ | $= \{ \lambda \in \mathbb{C} : \lvert\lambda\rvert = 1 \}$. |
| Sp(x) | spectrum of $x \in \mathcal{B}$ : $Sp(x) = \{ \lambda \in \mathbb{C}: \lambda\mathbf{e} - x \notin \mathcal{B}^* \}$ . |
| $V^-$ | topological closure of V, a subset in some topological space. |
| $V^\circ$ | set of inner points of V, a subset in some topological space. |
| $X(\mathcal{A})$ | Gelfand space of the maximal ideals of the commutative Banach algebra $\mathcal{A}$. |
| $\mathbb{Z}$ | the ring of integer numbers. |

# 1. Introduction

In a complex unital Banach algebra $\mathscr{B}$ you can define the exponential function exp: $\mathscr{B} \rightarrow \mathscr{B}$ equivalently by functional calculus or the following well known series, which converges relative to the norm topology :

$$\exp(x) = \sum_{k=0}^{\infty} 1/k!\ x^k \ , x \in \mathscr{B} .$$

This function (better: map) is Fréchet-analytic and its range $\exp(\mathscr{B})$ belongs to the principal component $\mathscr{B}^*_0$ of the group $\mathscr{B}^*$ of regular elements of $\mathscr{B}$ [BD]. In the early days (1947) of functional analysis J.v.Neumann, Halmos et al. [HP] recognized, that in general - in contrast to the case of a matrix algebra $M_n(\mathbb{C})$ - $\exp(\mathscr{B})$ and $\mathscr{B}^*_0$ are different.

This is true for $\mathscr{B} = BL(H)$ , the algebra of bounded linear operators on an infinite dimensional complex Hilbert space H, for which Deckard and Pearcy [DP] showed in 1967, that $\exp(\mathscr{B})$ lies neither open nor closed in $\mathscr{B}^*_0$ .

Pursuing Hille's theory [Hi] of linear ordinary differential equations ( in the complex domain) for functions attaining their values in $\mathscr{B}$, it turns out that properties of $\exp(\mathscr{B})$ define the limits to which extent known results in $M_n(\mathbb{C})$ carry over to the abstract case.

From this point of view the universal solvability of the classical Riemann-Hilbert problem over *non-compact* Riemann surfaces [Fo] relies essentially on the facts, that for the case $\mathscr{B} = M_n(\mathbb{C})$ we have $\mathscr{B}^*$ arcwise connected and $\mathscr{B}^* = \mathscr{B}^*_0 = \exp(\mathscr{B})$ .

This can be interpreted as an immediate consequence of:

**Theorem 1.1:**

*Let* $\mathscr{B}$ *be a complex unital Banach algebra ,* X *a connected, non-compact Riemann surface ,* $X' \subset X$ *a closed, discrete subset and* $\mu : \pi_1(X\backslash X') \rightarrow \mathscr{B}^*_0$ *a group homomorphism.*

*Define* $\mathfrak{C} := \{ \tau_x \in \pi_1(X\backslash X') \ : \ x \in X' \}$ *to be the set of homotopy classes of simple loops around exactly one point of* X', $p : Y \rightarrow X\backslash X'$ *the universal covering and* $\mathfrak{G} := \mathrm{Deck}(Y/X\backslash X') \cong \pi_1(X\backslash X')$ *the corresponding group of Deck transformations.*

*Suppose* <u>$\exp(\mathscr{B})$ *to be open in*</u> $\mathscr{B}$ *. Then the following statements are equivalent:*

*i)* $\mu(\mathfrak{C}) \subset \exp(\mathscr{B})$.

*ii) There is a meromorphic* $\mathscr{B}$ *-valued differential form* $\alpha$ *on* X*, holomorphic outside* X' *and with at most simple poles in* X'*, such that for a fundamental solution* $v_0 : Y \rightarrow \mathscr{B}^*_0$ *of the differential equation*

(*) $dv = p^*\alpha\, v$ *on* Y

*we have :* $\mathfrak{g}^* v_0 = v_0 \mu(\mathfrak{g}^{-1}) \ \forall\, \mathfrak{g} \in \mathfrak{G}$ .

[Al] proves the implication i ⇒ ii starting with the ideas of Röhrl, Grauert and Bungart for an arbitrary $\mathscr{B}$ . Due to a hint of Behnke and Stein relating to the simple topology of a non-compact Riemann surface, Röhrl can avoid the powerful Oka-principle in the case $\mathscr{B} = M_n(\mathbb{C})$; Grauert's daughter [Pe] extends this simpler argument to arbitrary $\mathscr{B}$ .
Theorem 1.1 now states conversely, that the opposite implication ii ⇒ i holds for an algebra $\mathscr{B}$ , which for example is commutative or in which each element has a totally disconnected spectrum, thus including the finite dimensional case.
It would be interesting to know of an algebra $\mathscr{B}$, in which $\exp(\mathscr{B})$ lies strictly but open in $\mathscr{B}^*_0$. Only for such $\mathscr{B}$ condition i is really distinctive.

Hille [Hi] likes to consider examples in $\mathscr{B} = BL(\mathscr{X})$, the Banach algebra of bounded linear operators on $\mathscr{X} := C[a,b]$, the Banach algebra of continuous complex-valued functions on the real interval [a,b], taken with the norm of uniform convergence.
If we define $A \in \mathscr{B}$ by setting $Af(t) := tf(t)$, $f \in \mathscr{X}$, $t \in [a,b]$, we can prove that $\exp(2\pi i A/(b-a))$ lies on the boundary of $\exp(\mathscr{B})$ relative to $\mathscr{B}^*_0$ , showing that in this case $\exp(\mathscr{B})$ is not open.
In contrast to the sophisticated constructions in [DP] for the Hilbert space case, the proof of our result remains entirely in the context of $\mathscr{B}$-valued differential equations.
We have to consider not only the differential equation (*) but also some distinguished local solutions of the differential equation
(**) $dw = p^*\alpha\, w - w\, p^*\alpha$ on Y ,
encountered already earlier - but more or less disguised - in [De], [Ga] or [Sch] .

It will suffice to treat a simple differential equation (*) ( see below in this paragraph) in the algebra $\mathscr{B} = \{(A - \frac{1}{2}(a+b)\,\mathbf{id})^2\}^c$ , which one could call - abusing a term in [Au] - *somewhat commutative* (as it admits of a family of finite-dimensional (even two-dimensional) irreducible representations separating the points of $\{(A - \frac{1}{2}(a+b)\,\mathbf{id})^2\}^c$) , but which is *not commutative enough to assure* $\mathscr{B}^*_0 = \exp(\mathscr{B})$ .

If we consider the equation (*) on an arbitrary Riemann surface X where the differential form $\alpha$ has at most a discrete closed set of singularities X', Hille calls simple poles of $\alpha$ *regular singular points* of the differential equation (*) , whereas in the case of $\mathscr{B} = M_n(\mathbb{C})$ he also refers to the German notion: *Stelle der Bestimmtheit*.
During the last decades these notions were not used very consistently, and the situation becomes worse, if systems of differential equations are considered, which stem from $n^{th}$ order linear differential equations.

Following the clarification of Anosov and Bolibruch [AB] we call simple poles of $\alpha$ - as we did in [Al] - *Fuchsian singularities*, while *regular singular points* of $\alpha$ are defined through a certain polynomial growth property of local solutions of (*) ; for details see [AB]. In a general Banach algebra $\mathscr{B}$ we get for any isolated singularity of $\alpha$: Fuchsian $\Rightarrow$ regular (cf. [Hi], p. 229, which for $\mathscr{B} = M_n(\mathbb{C})$ goes back to G.D. Birkhoff ).
But in $\mathscr{B} = M_n(\mathbb{C})$ the German notion *Stelle der Bestimmtheit* (going back to L.Fuchs for $n^{th}$ order linear differential equations) refers originally to the fact - in this form first recognized by Poincaré - that a fundamental solution of (*) locally and modulo logarithmic branching (described by a factor $\zeta^M$ with $M \in \mathscr{B}$) can be represented by a Laurent series with a finite principal part around the singularity of $\alpha$ in question.
For $\mathscr{B} = M_n(\mathbb{C})$ and an isolated singularity of $\alpha$ we have: Fuchsian $\Rightarrow$ Stelle der Bestimmtheit (cf. [Fo], p. 82).
In a general Banach algebra $\mathscr{B}$ again we obviously have for any isolated singularity of $\alpha$: Stelle der Bestimmtheit $\Rightarrow$ regular .

From the example below we can conclude that the opposite implication is not true for general Banach algebras:

**Theorem 1.2:**
*There are unital Banach algebras $\mathscr{B}$ with $\underline{\exp(\mathscr{B})}$ not open in $\mathscr{B}^*_0$ and in $\mathscr{B}$ a Fuchsian differential equation (*)(over $\mathbb{P}_1$), such that the factor of automorphy of a fundamental solution at at least one singularity of the differential equation does not belong to* $\exp(\mathscr{B})$ .

The cohomological approach of Röhrl in solving the Riemann-Hilbert problem (appropriately formulated) in so far as it concerns the nature of the singularities of $\alpha$ (it is exactly here, where the distinction between compact and non-compact surfaces becomes crucial, cf. [Al]) , essentially glues together local solutions. These local solutions in turn rely heavily on the functional equation of the exponential function, so that we are encouraged to conjecture, that equivalence of i and ii in Theorem 1.1. characterizes Banach algebras $\mathscr{B}$ with open $\exp(\mathscr{B})$ .

In this paper we will analyze the local question, under which circumstances the implication ii $\Rightarrow$ i in Theorem 1.1 becomes true in an arbitrary Banach algebra $\mathscr{B}$ .
The phenomena to be expected in the infinite dimensional (and non-commutative) case are illustrated in the following example (cf. section 7):

In $\mathscr{B}' = BL(\mathscr{X})$, $\mathscr{X} = C[-1,1]$ let $A \in \mathscr{B}'$ be the left multiplication with the identity on $[-1,1]$ and the operator N be defined by $N(f)(t) := f(-t)$, $f \in \mathscr{X}$, $t \in [-1,1]$ and consider the differential equation :

$$\zeta w'(\zeta) = ( A + N\, \zeta/(1-\zeta) )\, w(\zeta)\,, \zeta \in E^* = \{ \zeta \in \mathbb{C} : 0 < |\zeta| < 1\} \;.$$

Since $AN = -NA$ we can restrict our considerations to the algebra $\mathscr{B} = \{A^2\}^c$ of operators of $\mathscr{B}'$ which commute with $A^2$. But then in this smaller algebra there exists <u>no</u> fundamental solution of the form $w(\zeta) = H(\zeta)\, \zeta^M$, $M \in \mathscr{B}$, with a single-valued holomorphic function $H : E^* \to \mathscr{B}$ ; the cause of this are <u>not</u> the integers in the spectrum of the commutator $C_A$, $Sp(C_A) = [-2,2]$, but the fact that $\exp(2\pi iA)$ lies on the boundary of $\exp(\mathscr{B})$ relative to $\mathscr{B}^*_0$. If we consider this differential equation at the singular point $\infty$, we get:

$$\zeta w'(\zeta) = ( (N-A) + N\, \zeta/(1-\zeta) )\, w(\zeta)\,, \zeta \in E^* \,.$$

Here we have a fundamental solution of the form $w(\zeta) = H(\zeta)\, \zeta^{N-A}$ with H single-valued and holomorphic in E and $H(0) = \mathbf{id}_{\mathscr{B}}$, though $Sp(C_{N-A}) = \{0\} \cup [-2\sqrt{2}, -2] \cup [2, 2\sqrt{2}]$ obviously contains integers different from 0. Using the above cited result of Hille's we see that of the three Fuchsian singularities 0,1 and $\infty$ of this differential equation all are regular but only two are *Stellen der Bestimmtheit*.

If one tries to show by an example of the same kind (cf. section 9) that exp(BL(H)) is not open for any infinite-dimensional separable complex Hilbert space H, one should have to look for a non-normal operator $T \in BL(H)$, such that $\{T\}^c$ contains only few or better no non-trivial idempotents.

Starting point of our investigations however was the innocent looking differential equation (cf. section 8) :

$$\zeta w'(\zeta) = ( A + J\, \zeta^m )\, w(\zeta)\,, \zeta \in E^* \,,$$

in $\mathscr{B} = BL(\mathscr{X})$ with A as above. $J \in \mathscr{B}$ is defined by $J(f)(t) = \int_{[0,t]} f(s)ds$, $f \in \mathscr{X}$, $t \in [-1,1]$, and m is any fixed natural number. Hille ([Hi], p. 238) considers the case m = 1 and in vain tries to find a solution of the form

$(F_n)$ $w(\zeta) = ( H_0(\zeta) + \log(\zeta)\, H_1(\zeta) + (\log(\zeta))^2 H_2(\zeta) + \dots + (\log(\zeta))^{n-1} H_{n-1}(\zeta) )\, \zeta^A$,

where the $\mathscr{B}$-valued functions $H_k$ are holomorphic in E with $H_0(0) = (n-1)!\, \mathbf{id}$. This form of solution is suggested by the approach of Frobenius in the matrix case, but his arguments - as adapted by Hille - will not work here since the set of singularities of the resolvent of $C_A$

consists of the whole intervall $[-2,2]$ . Nevertheless we have for any $m \in \mathbb{N}$ , $m > 1$ , a fundamental solution w of the form $(F_1)$: that is $w(\zeta) = H(\zeta)\ \zeta^A$ with H single-valued and holomorphic in E, $H(0) = \mathbf{id}$.
This in contrast to the fact, that for $m = 1$ there is at least no fundamental solution of the form $(F_1)$, so Hille's question essentially remains open.

However, the existence of fundamental solutions of the form $(F_n)$ in a very special case can easily be used to obtain for example:

**Theorem 1.3:** *In a complex unital Banach algebra* $\mathscr{B}$ *with unit* $\mathbf{e}$ *, where* <u>*each element has its spectrum totally disconnected*</u>*, let* $f : E \to \mathscr{B}$ *be holomorphic .*
*Then there are*

*i)* $N \in \mathbb{Z}$ *,* $0 \le N$ *,*

*ii)* $n_i \in \mathbb{Z}$ *,* $|n_i| > 0$ *,* $i = 0, \ldots N$ *,*

*iii) a family of* N+1 *pairwise orthogonal idempotents* $p_i$ *from* $\{f(0)\}^c$ *which sum up to* $\mathbf{e}$ *,*

*iv) a holomorphic function* $h : E \to \mathscr{B}^*$ *with* $h(0) = \mathbf{e}$,

*and finally*

*v)* $y \in \mathscr{B}$ *,*

*such that*

$$w(\zeta) := h(\zeta)\ (\sum \zeta^{n_i}\ p_i)\ \zeta^{y}\ ,\ \zeta \in E\backslash[-1,0]\ ,$$

*is a fundamental solution of the differential equation:*

$$\zeta\, w'(\zeta) = f(\zeta)\, w(\zeta)\ .$$

In general we consider the differential equation (*) locally in the neighbourhood of a simple pole $\zeta_j$ of $\alpha$, i.e. essentially over the punctured unit disk E* . The corresponding differential equation (**) possesses a set of solutions, single-valued in E*, which necessarily have at most a pole in 0. The subset $\mathfrak{h}$ of these solutions, which can be extended analytically into 0 corresponds to a closed subalgebra $\mathscr{A}$ of $\{q_j\}^c$ , $q_j \in \mathscr{A}$, where $q_j$ is the factor of automorphy of a normalized local fundamental solution of (*) . If we name $a_j$ the residue of $\alpha$ at $\zeta_j$, so we further get a continuous morphism of Banach algebras $\Gamma_j : \mathscr{A} \to \{a_j\}^c$ with $\Gamma_j(q_j) = \exp(2\pi i a_j)$.
We have $Sp_{\mathscr{A}}(q_j) = Sp_{\mathscr{B}}(\exp(2\pi i a_j))$ and the integer eigenvalues of the commutator corresponding to $a_j$ mainly determine the size resp. structure of $\mathscr{A}$ resp. of the kernel of $\Gamma_j$ . Whether $q_j$ belongs to $\exp(\mathscr{B})$ , depends - if not upon $\mathscr{B}$- then upon $\{a_j\}^c$ or the range of $\Gamma_j$
.

In $\mathfrak{H}$ there exists - as local invariant of the differential equation (*) - a distinguished solution of (**) , which we call the *transformation of monodromy* ( in the spirit of [De] ) and which provides useful information as well, as it parametrizes the factors of automorphy of all normalized fundamental solutions.
In addition, not only increasing but also shrinking the Banach algebra $\mathscr{B}$ may clarify a concrete situation.

As is commonly experienced in the general theory of Banach algebras the individual results are simple to state and the proofs finally turn out to be quite elementary, but application to concrete problems requires subtle combinations of many of them. As *leitmotiv* we see here the necessity of a careful handling of commutants and of a clear distinction of the spectrum of an operator from the subset of its eigenvalues, to which it partly may reduce when we consider the restriction of the operator to an appropriate invariant subspace.

## 2. Stating the Problem

The local nature of the problem as sketched above allows us to confine ourselves to simplest Riemann surfaces.

Let $E := \{ \zeta \in \mathbb{C} : |\zeta| < 1\}$, $E^* = E \setminus \{0\}$, $H := \{ \sigma \in \mathbb{C} : \mathrm{Rea}\,\sigma < 0 \}$, $p := \exp : H \to E^*$ the universal covering. The group of Deck transformations relative to p $\mathfrak{G} := \mathrm{Deck}(H/E^*)$ is generated by $\mathfrak{g}_0$, $\mathfrak{g}_0(\sigma) := \sigma + 2\pi i$, thus is infinitely cyclic.

Let $\mathscr{B}$ be an arbitrary - but fixed - non commutative complex Banach algebra with unit **e**.

For $U \subset \mathbb{C}$ open and a map (or function) $f : U \to \mathscr{B}$ a well known result of Dunford [HP] establishes equivalence between the following statements:

i) f is complex differentiable at each point of U (in the sense of existence of the differential quotient and limits in norm)

ii) f is weakly complex differentiable at each point of U (in the sense that for each linear bounded functional $\varphi$ on $\mathscr{B}$ the complex-valued function $\varphi \circ f$ is complex differentiable in the usual sense)

iii) locally in U f can be developed in a Taylor series with coefficients from $\mathscr{B}$, which is (locally uniformly) convergent in norm and absolutely convergent.

We call a function f satisfying one and thus all of these condtions *holomorphic*. Holomorphic functions are clearly continuous.

Let $f : E \to \mathscr{B}$ be holomorphic, $f(0) =: a_0 \in \mathscr{B}$, $a_0 \neq 0$, i.e. the series

$$f(\zeta) = \sum_{k=0}^{\infty} a_k \zeta^k, \quad a_k \in \mathscr{B},$$

converges compactly (that is uniformly on compact subsets) in E relative to the norm of $\mathscr{B}$.

Then

$$\alpha := f(\zeta) / \zeta \, d\zeta$$

defines a meromorphic $\mathscr{B}$-valued differential form on E with a pole of first order in 0 .

These last notions can be used rather intuitively here, for details we refer to [Al].

For $\sigma_0 \in H$ we have a unique fundamental solution (cf. Lemma 3.1) $v_0 : H \to \mathscr{B}^*_0$ of the differential equation

(*) $dv = p^*\alpha \; v$

with $v_0(\sigma_0) = \mathbf{e}$ ;

that is in the coordinate $\sigma$ on H :

$$v_0'(\sigma) = f(e^{\sigma}).v_0(\sigma), \quad v_0(\sigma_0) = \mathbf{e} ,$$

where the prime ' stands for the complex derivative in the sense of i above.
For simplicity we call a fundamental solution *normalized,* if it attains the value **e** ([Ga]).

Obviously $(\mathfrak{g}_0^{-1})^* v_0 := v_0 \circ \mathfrak{g}_0$ is another fundamental solution of (*), i.e. there is a unique $q_0 \in \mathscr{B}^*_0$ with $(\mathfrak{g}_0^{-1})^* v_0 = v_0 q_0$.
We call $q_0$ the *factor of automorphy* of the solution $v_0$ of (*) .

Since $\mathscr{B}^*_0$ is a normal subgroup of $\mathscr{B}^*$ , the factor of automorphy of any fundamental solution of (*) - not necessarily normalized - belongs to $\mathscr{B}^*_0$ .

First of all - in the spirit of Poincaré - we would like to inquire under what conditions $q_0$ belongs to $\exp(\mathscr{B})$ , i.e..
**(P1)** *when does a* $y_0 \in \mathscr{B}$ *exist with:* $q_0 = \exp(2\pi i y_0)$ ?
Because then we have $v_0(\sigma) \exp(-y_0(\sigma-\sigma_0))$ invariant under $\mathfrak{G}$, i.e. there is a holomorphic function $m : E^* \to \mathscr{B}^*$ with

$$v_0(\sigma) = p^*m(\sigma) \ \exp(y_0(\sigma-\sigma_0)) = m(e^{\sigma}) \ \exp(y_0(\sigma-\sigma_0)) .$$

Since $\exp(\mathscr{B})$ is invariant under conjugation with elements from $\mathscr{B}^*$, it will suffice to answer the question P1 for an arbitrary factor of automorphy.
Consequently we are interested in the behavior of m in 0: we shall see ( Proposition 6.2) that if P1 has a positive answer, then the function m has at most a pole in 0 .
We may ask especially :
**(P2)** *when does* m *admit of an analytic continuation into 0 which is regular there ?*
Again a positive or negative answer to P2 does not depend on a specific fundamental solution (cf. Proposition 6.4) , so we adopt the settings of this paragraph as a basic assumption for the rest of this paper. Especially $\sigma_0$ and therefore $v_0$ are supposed to be fixed.

P2 is by Proposition 6.4 a special case (n = 1) of:
**(P3)** *are there a* $n \in \mathbb{N}$ *and* n $\mathscr{B}$*-valued functions* $h_0, \dots, h_{n-1}$ *holomorphic on* E *with* $h_0(0) = (n-1)!\mathbf{e}$ , $h_k(0) = 0$, k= 1,...,n-1, *and* $h_{n-1}$ *not identically zero such that*
$v(\sigma) := (h_0(e^{\sigma}) + \sigma h_1(e^{\sigma}) + \dots + \sigma^{n-1}h_{n-1}(e^{\sigma})) \exp(a_0\sigma)$ , $\sigma \in H$,
*defines a fundamental solution of (*) ?*
We call a fundamental solution of this form a *Frobenius solution ( of type n )* .

Suppose the set $\mathfrak{f} := \{f(\zeta) : \zeta \in E\}$ to be commutative, then we have
$v_0(\sigma) = \exp(\int_C p^*f(\xi)d\xi)$
with a suitable curve C in H joining $\sigma, \sigma_0$ , that is: $q_0 = \exp(2\pi i a_0)$ .
All considerations can take place in a commutative, closed subalgebra of $\mathscr{B}$ containing $\mathfrak{f}$ and **e** , for example $\mathfrak{f}^{cc}$ (cf. 3).

With $h(\zeta) := (f(\zeta) - a_0)/\zeta$ one gets:
$v_0(\sigma) = p^* m_0(\sigma)\exp(a_0(\sigma-\sigma_0))$, with $m_0$ the fundamental solution of $m'(\zeta) = h(\zeta)\ m(\zeta)$ with $m_0(\zeta_0) = \mathbf{e}$, $\zeta_0=\exp(\sigma_0)$. $m_0$ is holomorphic and regular in 0 ( cf. Lemma 3.1).

In the sequel we thus suppose $\mathfrak{F}$ to be non-commutative.

Note that positive answers to *both* questions P1 and P2 necessarily require the differential form $\alpha$ to have at most a pole of order one in 0 .
Some of our considerations remain valid for poles of higher order (e.g. Proposition 5.3), whereas some other (e.g. Lemma 3.4, Proposition 5.2) definitely fail (cf. example 5.7).

In the case of a matrix algebra $\mathscr{B} = M_n(\mathbb{C})$ P1 always has a positive answer; conversely for any $q_0 \in \mathscr{B}^*$ there is a differential equation (*) with $q_0$ as factor of automorphy of a fundamental solution:
If we have $q_0 = \exp(2\pi i y_0)$ with $y_0 \in \mathscr{B}$, let $f(\zeta) := y_0$. We see: $f(\zeta) = f(-\zeta)$ .

For general $\mathscr{B}$ let us note a partial reverse:

**Proposition 2.1:** *Let* $k>1$ *be a natural number,* $\eta := \exp(2\pi i/k)$ . *If we have* $f(\zeta) = f(\eta\zeta)$ *for* $\zeta \in E$ *(i.e. all Taylor coefficients of* $f$ *around* 0 *vanish, whose index is not a multiple of* $k$ *), then* $q_0$ *has a* $k^{th}$ *root in* $\mathscr{B}^*_0$ .

**Proof:**
For $\sigma \in H$ let $u(\sigma) := v_0(\sigma + 2\pi i/k)$ ; then we have
$u'(\sigma) = v_0'(\sigma + 2\pi i/k) = f(\eta e^{\sigma}).v_0(\sigma + 2\pi i/k) = f(e^{\sigma})\, u(\sigma)$ ,
i.e. $u$ is equally a fundamental solution of (*).
Consequently there is a $x \in \mathscr{B}^*_0$ with $u(\sigma) = v_0(\sigma)\, x$ .
It follows for $\sigma \in H$
$u(\sigma + 2\pi i/k) = v_0(\sigma + 2\pi i/k)\, x = u(\sigma)\, x$, which for $m > 1$ yields
$u(\sigma + 2 m \pi i/k) = u(\sigma + 2 (m-1) \pi i/k)\, x = u(\sigma + 2 (m-2) \pi i/k)\, x^2 = .... = u(\sigma)\, x^m$ .
Thus:
$v_0(\sigma)\ q_0 = v_0(\sigma + 2\pi i/k + 2(k-1) \pi i/k) = u(\sigma + 2(k-1) \pi i/k) = u(\sigma)\, x^{k-1} = v_0(\sigma)\, x^k$ .
q.e.d.

In short we call a function $f$ *k-invariant,* if it satisfies the assumption of Proposition 2.1 .

For general $\mathscr{B}$ but under rather severe conditions on $\mathrm{Sp}(a_0)$ - which are automatically satisfied in a matrix algebra $M_n(\mathbb{C})$ - Hille shows( [Hi], p. 234 ff) that (*) has a Frobenius solution. Let us consider the converse:

**Proposition 2.2:** *Suppose we had a Frobenius solution* $v$ *of type* $n$ *of (*), i.e. there are* $n \in \mathbb{N}$ *and* $h_0, \ldots, h_{n-1} \in \mathscr{O}^{\mathscr{B}}(E)$ *with* $h_0(0) = (n-1)!\mathbf{e}$ , $h_k(0) = 0$, $k = 1,\ldots, n-1$ , $h_{n-1}$ *not identically zero and* $v(\sigma) = (h_0(e^{\sigma}) + \sigma\, h_1(e^{\sigma}) + \ldots + \sigma^{n-1}h_{n-1}(e^{\sigma}))\exp(a_0\sigma)$ , $\sigma \in H$ .

*i) The functions* $h_k$ *are solutions of the differential equations* $(\zeta \in E\ )$*:*

$$\zeta h_k'(\zeta) = f(\zeta)h_k(\zeta) - h_k(\zeta)\, a_0 - (k+1)h_{k+1}(\zeta)\ , \ k = 0,\ldots,n-2$$

$$\zeta h_{n-1}'(\zeta) = f(\zeta)h_{n-1}(\zeta) - h_{n-1}(\zeta)\, a_0 \ .$$

*ii) If we denote by* $n_k$ *the order of* $h_k$ *in* 0*, then* $n_0 = 0$ , $1 \le n_k \le n_{k+1}$ , $k = 1,\ldots, n-2$, $n_{n-1} \in \mathrm{PSp}(\mathrm{C}a_0)$.

*iii) If* $\mathrm{PSp}(\ C_{f(0)}\ ) \cap \mathbb{N} = \varnothing$ *or if* $q_0^{\,m} = \mathbf{e}$ *for some* $m \in \mathbb{N}$ *then* $n = 1$ .

*iv) If* $\mathrm{PSp}(\ C_{f(0)}\ ) \cap \mathbb{N} = \{1\}$ *then* $n = 1$ *or* $(\mathrm{C}a_0 - \mathbf{id}\ )^k\,(c_1) \neq 0$ for $k = 0,\ldots, n-2$ , $(\mathrm{C}a_0 - \mathbf{id}\ )^{n-1}\,(c_1) = 0$, *where* $c_1$ *is the leading coefficient of* $h_1$ .

**Proof:**

i) It is easy to see (by induction on n and using the substitution $\sigma \to \sigma + 2\pi i$ ) that an expression of the form $h_0(e^{\sigma}) + \sigma\, h_1(e^{\sigma}) + \ldots + \sigma^{n-1}h_{n-1}(e^{\sigma})$ vanishes identically on H if and only if all the functions $h_k$ are identically zero.

This rather formal argument applies also to the case, where we suppose the functions $h_k$ only to be single-valued in E* such that $h_{n-1}$ does not vanish identically.

So the assertion is immediate from (*) .

ii) follows from the first n-1 differential equations. That means also that v is uniquely determinded by $h_0$ :

Write $h_k(\zeta) = \zeta^{\,n_k} g_k(\zeta)$ with $c_k := g_k(0) \neq 0$. Then

$$n_k\, g_k(\zeta) + \zeta g_k'(\zeta) = f(\zeta)g_k(\zeta) - g_k(\zeta)\, a_0 - (k+1)\ \zeta^{\,n_{k+1} - n_k}\ g_{k+1}(\zeta)\ , \ k = 0,\ldots,n-2$$

$$n_{n-1}\, g_{n-1}(\zeta) + \zeta g_{n-1}'(\zeta) = f(\zeta)g_{n-1}(\zeta) - g_{n-1}(\zeta)\, a_0 \ .$$

So clearly $n_k \le n_{k+1}$ , $k = 0,\ldots, n-2$, and $n_{n-1}$ belongs to $\mathrm{PSp}(\mathrm{C}a_0)$ , $c_{n-1}$ is nilpotent.

iii) If $\mathrm{PSp}(\ C_{f(0)}\ ) \cap \mathbb{N} = \varnothing$ then the last differential equation has only 0 as solution of order > 0 , so $h_{n-1} = 0$ (cf. Proposition 6.5). Caution: The opposite implication is not true (cf. Example 6.16) .

Suppose now we had $q_0^{\,m} = \mathbf{e}$ then all fundamental solutions of (*) are invariant under $(\mathfrak{g}_0^{\,-m})^*$ .

Write $u(\sigma) := h_0(e^{\sigma}) + \sigma\, h_1(e^{\sigma}) + \ldots + \sigma^{n-1}h_{n-1}(e^{\sigma})$ then we have for $\sigma \in H$:

$v(\sigma) = u(\sigma)\ \exp(a_0\sigma) = u(\sigma + 2\pi i m\ )\exp(2\pi i m a_0)\ \ \exp(a_0\sigma)$ , so $u(\sigma) = u(\sigma + 2\pi i m)\exp(2\pi i m a_0)$ .

But $u(\sigma)$ and $u(\sigma + 2\pi i m)$ tend to the same limit ( $(n-1)!\mathbf{e}$) if $\sigma \to -\infty$, hence $\exp(2\pi i m a_0) = \mathbf{e}$ (cf. Proposition 5.1) , u is itself invariant under $(\mathfrak{g}_0^{-m})^*$ . If $n > 1$ this would require $h_{n-1} = 0$ .

iv) By assumption we have $PSp( C_{f(0)} ) \cap \mathbb{N} = \{1\}$ ,thus $n_k = 1$ for $k = 1,\dots,n-2$ .

It follows $c_{k+1} = 1/(k+1)\,({}^{C}a_0 - \mathbf{id})(c_k)$ for $k = 1,\dots,n-2$ , hence

$c_{n-1} = 1/(n-1)!\ \ ({}^{C}a_0 - \mathbf{id})^{n-2}(c_1)$ . We suppose $n > 1$ .

It follows:

$({}^{C}a_0 - \mathbf{id})^k (c_1) \neq 0$ for $k = 0,\dots, n-2$ , $({}^{C}a_0 - \mathbf{id})^{n-1} (c_1) = 0$.

q.e.d.

Before we can start to tackle the problems P1 and P2, we have to prepare ourselves with some basic facts and elementary results from the theory of Banach algebras. We pick up the actual questions again in section 5, so the reader mainly interested in differential equations may skip sections 3 and 4 and come back to them later when necessary.

# 3. Preliminaries

In this section we collect some basic results used later to streamline the main arguments and to fix notation. As said we try to maintain the conventions of section 2. For basic notions from the theory of Banach algebras - like ideals, representations, semi-simplicity etc. - we refer to [Au] or [BD] .

## 3.1 Notation

The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ stand for the natural, integer, rational, real and complex numbers respectively, each provided with the ususal algebraic or topological structure.
$BL(\mathscr{B})$ is defined to be the Banach algebra of bounded, linear operators from $\mathscr{B}$ into itself, provided with the usual operator norm.
For a compact subset $K$ of $\mathbb{C}$ let $\mathrm{diam}(K) := \sup\{|\lambda - \mu| : \lambda,\mu \in K\}$ and
$\mathrm{Rea}(K) := \{\mathrm{Rea}(\mu) : \mu \in K\}$ .
For $\mathscr{A} \subset \mathscr{B}$ , a not necessarily closed subalgebra with the same unit **e** as $\mathscr{B}$ and $x \in \mathscr{A}$ let $\mathrm{Sp}_{\mathscr{A}}(x)$ denote the spectrum of $x$ relative to $\mathscr{A}$:
$\mathrm{Sp}_{\mathscr{A}}(x) := \{\lambda \in \mathbb{C} : \lambda\mathbf{e} - x \notin \mathscr{A}^*\}$, $\mathrm{Sp}(x) := \mathrm{Sp}_{\mathscr{B}}(x)$ .
A subalgebra $\mathscr{A} \subset \mathscr{B}$ is called *full*, if it contains the unit of $\mathscr{B}$ and if it closed under taking inverses, i.e. $\mathscr{A}^* = \mathscr{A} \cap \mathscr{B}^*$, [Bou ].
Thus for a full subalgebra $\mathscr{A} \subset \mathscr{B}$ and $x \in \mathscr{A}$ we have $\mathrm{Sp}_{\mathscr{A}}(x) = \mathrm{Sp}_{\mathscr{B}}(x)$ .

The spectral radius of $x \in \mathscr{B}$ is denoted by $\rho$ : $\rho(x) = \max\{|\lambda| : \lambda \in \mathrm{Sp}(x)\}$ , the resolvent set of $x$ by $R(x)$ and its unbounded component by $R_0(x)$ .

If $\mathscr{B}$ happens to be $BL(X)$ for some (complex) Banach space $X$ we denote for $x \in \mathscr{B}$ by $\mathrm{PSp}(x)$ the point spectrum of $x$, that is the set of all eigenvalues of $x$, $\mathrm{PSp}(x) \subset \mathrm{Sp}(x)$ .

For $a \in \mathscr{B}$ let $L_a, R_a \in BL(\mathscr{B})$ be defined by $L_a(x) = ax$ , $R_a(x) = xa$ , $x \in \mathscr{B}$ . The commutator for a is simply $C_a := L_a - R_a$ .
For $a,b \in \mathscr{B}$ one easily checks: $C_aC_b - C_bC_a = {}^{C}C_a(b)$ .
The norm $\|.\|$ on $\mathscr{B}$ is in general supposed to satisfy $\|\mathbf{e}\| = 1$ .
That means, that the left regular representation of $\mathscr{B}$ on itself : $L(a) := L_a$ , $a \in \mathscr{B}$ , is an isometric injection $L: \mathscr{B} \to BL(\mathscr{B})$.

For $x \in \mathscr{B}^*$ let $\alpha_x$ denote the inner automorphism of $\mathscr{B}$ corresponding to x:
$\alpha_x(y) := xyx^{-1}$ , $\forall\ y \in \mathscr{B}$ .

### 3.2 Existence, Uniqueness and Growth of Solutions

**Lemma 3.1:** *Let* $F : E \to BL(\mathscr{B})$ *and* $f : E \to \mathscr{B}$ *be holomorphic.*
*i) For each* $b_0 \in \mathscr{B}$ *there is a unique holomorphic solution* $h_0 : E \to \mathscr{B}$ *of the differential equation*
*(+)* $g'(\zeta) = F(\zeta)(g(\zeta))$ *with* $h_0(0) = b_0,\ \zeta \in E$.
*ii) We have*

$$h_0(\zeta) = V_0(\zeta)(b_0), \quad \zeta \in E,$$

*with* $V_0\colon E \to BL(\mathscr{B})^*{}_0$ *the fundamental solution of*
*(++)* $V'(\zeta) = F(\zeta)V(\zeta)\ ,\ \zeta \in E,$
*with* $V_0(0) = \mathbf{id}_{\mathscr{B}}$ .
*iii) For each* $b_1 \in \mathscr{B}$ *there are unique holomorphic solutions* $h_k : E \to \mathscr{B}$ , $k = 1,2$ , *of the differential equations*
*(a)* $g'(\zeta) = f(\zeta)g(\zeta)$ *with* $h_1(0) = b_1,\ \zeta \in E$
*(b)* $g'(\zeta) = g(\zeta)f(\zeta)$ *with* $h_2(0) = b_1,\ \zeta \in E$ .

**Remark:** *iii a is precisely the local version of the statement cited under 2. The combination of iii a and b provides ,following Coppel (cf. [Hi]), a simple argument to show that in iii a or b all values of* $h_k$ *are regular if and only if this is true for the value at only one point of* $E$ . *In this case we call* $h_k$ *a* fundamental solution .

**Proof:**
i) Since the (linear) elements of $BL(\mathscr{B})$ commute with complex differentiation and the Taylor series representating F and f are absolutely convergent, it is straightforward to apply Cauchy's method of majorants.
iii) apply i to $F := L_f$ resp. $F := R_f$ .
ii) follows from iii (applied to $BL(\mathscr{B})$ ) and uniqueness by i. q.e.d.

In [Hi] or [Al] it is shown that the statement of iii remains valid if one substitutes E by a simply connected Riemann surface ( f replaced by a *holomorphic* $\mathscr{B}$-valued differential form and ' by d) . From ii we thus get - analogously replacing F by a $BL(\mathscr{B})$-valued differential form - that equally local solutions of i can be continued analytically throughout a simply connected Riemann surface .

With real differentiability defined analogously to the complex one, its fundamental connection with the (Riemann-) integral yields:

**Lemma 3.2 (Gronwall):** *Let* $\varphi : [0,1] \to \mathscr{B}$ *be continuously differentiable and suppose the existence of* $M>0$, $0 \leq N$ *such that for* $t \in [0,1]$ *we have:*

$$\| \varphi'(t) \| \leq M \|\varphi(t)\| + N.$$

*Then for* $t \in [0,1]$:

$$\| \varphi(t)\| \leq (\| \varphi(0)\| + N)e^{Mt}.$$

**Proof:** Simple application of [Hi], Theorem 1.5.7, Corollary 2 . q.e.d.

Let $a,b \in \mathbb{R}$ with $b > a$ and $C[a,b]$ the (complex) Banach algebra of continuous complex-valued functions on $[a,b]$ with the max-norm. Analogously let $C_{\mathbb{R}}[a,b]$ be the (real) Banach algebra of continous real-valued functions on $[a,b]$ equally equipped with the max-norm. Then $C_{\mathbb{R}}[a,b] \subset C[a,b]$ is a closed real subalgebra.
If we have $T \in BL(C[a,b])$, such that $C_{\mathbb{R}}[a,b]$ is invariant under T, and we denote by $\mathfrak{T}$ the restriction of T to $C_{\mathbb{R}}[a,b]$ considered as element of $BL(C_{\mathbb{R}}[a,b])$, we conlude (cf. Lemma 3.9) for the corresponding operator norms: $\|\mathfrak{T}\| \leq \|T\|$.

**Lemma 3.3:** *Let* $k \in C_{\mathbb{R}}[a,b]$ *; then*

$$T(f)(t) := \int_{[t,\, b]} k(s)f(s)ds, \ \forall f \in C[a,b],\ t\in [a,b],$$

*defines a quasi-nilpotent operator* $T \in BL(C[a,b])$, *which leaves* $C_{\mathbb{R}}[a,b]$ *invariant.*

*The series* $\sum \|\mathfrak{T}^n\|$ *converges.*

*(It should be noticed, that in the definition of T the* *__lower__* *integration bound is variable.)*

**Proof:**

T is well defined, linear and bounded with $\|T\| \leq (b-a)\ \|k\|$.

Let $g(t) := \int_{[t,\, b]} k(s)ds$ for $t\in [a,b]$, i.e. $g \in C_{\mathbb{R}}[a,b]$ .

Then we have for $f \in C[a,b]$, $t\in [a,b]$ :

$$\begin{aligned} T^2(f)(t) &= \int_{[t,\, b]} k(s)\, T(f)(s)ds = \int_{[t,\, b]} k(s) \int_{[s,\, b]} k(u)f(u)du\, ds \\ &= g(t)\, T(f)(t) - \int_{[t,\, b]} g(s)k(s)f(s)ds, \end{aligned}$$

i.e. $T^2 = L_gT - TL_g$, with $L_g \in BL(C[a,b])$ .

T commutes with $TL_g - L_gT$ , the Theorem of Kleinecke and Shirokov ([Au], p.20) says that $TL_g - L_gT = - T^2$ is quasi-nilpotent (i.e. $\rho(T^2) = 0$ ), the Spectral Mapping Theorem [HP] yields $\rho(T) = 0$.

Consequently on BL(C[a,b] ) there is an equivalent algebra norm $\|.\|'$ with $\|T\|' < 1/2$ ( cf. [Au], p. 123), i.e. we have a $r > 0$ with

$\|\mathfrak{T}^n\| \le \|T^n\| \le r\|T^n\|' \le r(\|T\|')^n < r(1/2)^n$,

and this implies convergence of $\sum \|\mathfrak{T}^n\|$ . q.e.d.

In the sequel we don't need any growth properties of solutions of (*) , which go further than to apply Lemma 3.2 or the following statement, which is known for matrix-algebras, but usually is proved by methods not suitable for generalization (cf. [Fo], p. 82). Nevertheless it holds for arbitrary $\mathcal{B}$:

**Lemma 3.4 :** *Let* $h: E^* \to \mathcal{B}$ *be holomorphic in* $E^*$ *and suppose there is a* $M>0$ *with :*

$\|\zeta h'(\zeta)\| \le M\|h(\zeta)\|$ , $\zeta \in E^*$ .

*Then* h *has in* 0 *at most a pole of* $M^{th}$ *order .*

**Proof:** Let $M > 0$ satisfy the assumption and let

$0<r<1$, $0 \le \vartheta < 2\pi$ , $\zeta = re^{i\vartheta}$ . For arbitray but fixed $\vartheta$ we set:

$G(r) := h(re^{i\vartheta})$,

i.e.

$(d/dr)\, G(r) = h'(re^{i\vartheta})e^{i\vartheta}$ ,

$\|(d/dr)G(r)\| \le M/r\, \|G(r)\|$ ,

finally let:

$g(r) := \|G(r)\|$ .

If we fix $0 < r_0 < \rho < 1$ with $r_0 \le r \le \rho$ , we get:

$$G(r) = G(\rho) - \int_{[r,\rho]} G'(s)ds$$

$$0 \le g(r) \le g(\rho) + \int_{[r,\rho]} M/s\, g(s)ds.$$

Now [Hi], Theorem 1.5.6 yields:

$$(+) \qquad g(r) \le g(\rho)(\rho/r)^M, \quad r \in [r_0,\rho],$$

because $f(r) := g(\rho)(\rho/r)^M$ is fixed point of the map $\mathfrak{T} : C_{\mathbb{R}}[r_0,\rho] \to C_{\mathbb{R}}[r_0,\rho]$ defined by

$$\mathfrak{T}(f)(r) := g(\rho) + \int_{[r,\rho]} M/s\, f(s)ds.$$

The map $\mathbb{S} : C_{\mathbb{R}}[r_0,\rho\,] \to C_{\mathbb{R}}[r_0,\rho\,]$, $\mathbb{S}(f)(r) := \int_{[r,\,\rho\,]} M/s\; f(s)ds$

is positive, linear and bounded; Lemma 3.3 shows that all conditions of [Hi], Theorem 1.5.6 are satisfied.

The estimation (+) is independent of $\vartheta$ , $r_0$ is arbitrary, thus:

$$\|h(\zeta)\| \le \sup\{\|h(\xi)\| : |\xi|=\rho\}\,(\rho/|\zeta|)^M, \quad 0 < |\zeta| < \rho\ . \qquad \text{q.e.d.}$$

### 3.3 Commutants

For a subset M of $\mathscr{B}$ we denote by $M^c$ the (first) commutant (or centralizer) of M, that is the set of all elements of $\mathscr{B}$ , commuting with all elements of M.

$M^c$ is obviously a closed, full subalgebra of $\mathscr{B}$.

If M is commutative, then we get for the commutant of $M^c$: $M \subset M^{cc} \subset M^c$ .

For an arbitrary subset $M \subset \mathscr{B}$ we note:

$M \subset M^{cc} = M^{4c} = M^{6c} = \ldots.$

$M^c = M^{ccc} = M^{5c} = \ldots.$

**Lemma 3.5:**

*i) Let* $x \in \mathscr{B}^*$ , $y \in \mathscr{B}$ *with* $xy = yx$ . *Then the restriction of* $\alpha_x$ *to* $\{y\}^c$ *is a (Banach algebra) automorphism* $\alpha_x : \{y\}^c \to \{y\}^c$ .

*ii) Let* $x,y \in \mathscr{B}$ . *If there is a* $z \in \mathscr{B}^*$ *with* $y = zxz^{-1}$ , *then* $\alpha_z : \{x\}^c \to \{y\}^c$ *is a ( Banach algebra ) isomorphism.*

The proof is immediate.

Let for $x \in \mathscr{B}$ $M_x$ be a maximal commutative subset of $\mathscr{B}$ containing x ; for $M \subset \mathscr{B}$ let V(M) be the smallest (closed) full subalgebra of $\mathscr{B}$ containing M ( $V(x) := V(\{x\})$ ) . Then we get

**Lemma 3.6 :** *For* $x \in \mathscr{B}$ *we have*:

*i)* $\{x\}^c$ *is commutative* $\Leftrightarrow$ $\{x\}^{cc} = M_x = \{x\}^c$ .

*ii)* $V(x)^c = \{x\}^c$ .

**Proof:** i) $\Leftarrow$ is trivial.

Because of $M_x = M_x^{cc}$ in general, we get:

$\{x\} \subset \{x\}^{cc} \subset M_x \subset \{x\}^c$ .

If $\{x\}^c$ is commutative, it follows $M_x = \{x\}^c$ by maximality. Since $M_x$ is commutative, taking commutants yields:

$M_x \subset (M_x)^c = \{x\}^{cc} \subset M_x$ .

ii) Obviously: $V(x) \subset \{x\}^{cc}$ , i.e. $\{x\}^c \subset V(x)^c$ . On the other hand $x \in V(x)$: $V(x)^c \subset \{x\}^c$ . q.e.d.

The elements of a set $\{ p_i \in \mathscr{B} : i \in I \}$, I some index set, of idempotents in $\mathscr{B}$ are called *pairwise orthogonal,* if $p_i p_j = \delta_{ij} p_j$ , $i,j \in I$ ($\delta_{ij}$ = Kronecker delta).

**Lemma 3.7:** *Let* $a \in \mathscr{B}$ , $p_i \in \mathscr{B}$ , $i=1,\ldots,r$ *pairwise orthogonal idempotents with* $\sum p_i = \mathbf{e}$ *and* $n_i$, $i=1,\ldots r$ *integers* , $q := \sum n_i p_i$ .

*i ) If* $q \in \{a\}^c$ *then there are* $s \in \mathbb{N}$, $s \le r$ *and* $q_i \in \{a\}^c$ , $i=1,\ldots,s$ *pairwise orthogonal idempotents and integers* $m_i$ , $i=1,\ldots,s$ *with* $q = \sum m_i q_i$ *and* $\sum q_i = \mathbf{e}$ .

*ii)* $Sp(C_q)$ *is finite and the resolvent* $( \lambda\, \mathbf{id} - C_q )^{-1}$ *has at most poles in* $Sp(C_q)$.

**Proof:**

a) From $\exp(2\pi i q) = \mathbf{e}$ and $q \in \{a\}^c$, it follows that q is an integer linear combination of pairwise orthogonal idempotents from $\{a\}^c$ (cf. [BD], Proposition 8.11, p. 42); the Lemma essentially states that the number of idempotents remains bounded if the idempotents are taken from the smaller algebra $\{a\}^c$.

i) r = 1 is trivial, as is the case q = 0. Let $q \neq 0$.

ii) Let r = 2, i.e. with $p := p_1$, $q := p_2$, $m := n_1$ and $n := n_2$ let $u := mp + nq \in \{a\}^c$ .

au = ua then means :

$$map + naq = mpa + nqa$$

$$m(ap - pa) = n(qa - aq)$$

i.e. $-mpaq = nqaq - naq$ resp.

$$mqap = nqa - nqaq.$$

That is: $-mpaq = - npaq$ resp. $mqap = nqap$,

i.e. $m = n$ or ( $paq = 0$ and $qap = 0$).

Suppose, $m \neq n$, i.e. $paq = 0$ und $qap = 0$.

We get $qaq = aq$ and $qaq = qa$ , thus $q \in \{a\}^c$ and consequently $p \in \{a\}^c$ .

If $m = n$, then we have $u = m\,\mathbf{e}$ , because $\mathbf{e} = p + q \in \{a\}^c$ .

iii) Let $r > 2$, $q = \sum n_i p_i \in \{a\}^c$ , $\sum p_i = \mathbf{e}$ , $p_i$ pairwise orthogonal idempotents.

If we apply induction over r, the start is provided by ii. Let the statement be true for all integer linear combinations of pairwise orthogonal idempotents with s items, $s < r$ .

From $a q = q a$ we get:

$$n_k p_i a p_k = n_i p_i a p_k , \quad i,k = 1, \ldots, r.$$

If for i,k different $p_i a p_k \neq 0$, then $n_k = n_i$ . Collecting the $p_i$ belonging to equal $n_i$ , we get a linear combination of s items , s < r, and the statement follows by induction hypothesis.

In the opposite case $p_i a p_k = 0$ , $\forall$ i,k = 1,..., r with $i \neq k$.

From $a = \mathbf{e}\, a\, \mathbf{e} = \sum_{i,j} p_i \; a\, p_j = \sum_i p_i \; a\, p_i$

we get $a p_k = p_k a p_k = p_k a$ , k = 1,...,r .

b) $C_q$ belongs to the closed subalgebra of $BL(\mathscr{B})$ generated by $L_{p_i}$, $R_{p_j}$ , i,j = 1,....,r , which is clearly unital, finite dimensional, hence closed and full .

q.e.d.

The following lemma collects some elementary facts useful in the examples at the end of this paper.

**Lemma 3.8:** *Let* $\mathscr{B}$ *be a complex Banach algebra with unit* $\mathbf{e}$ *und* $a \in \mathscr{B}$ .

*Then :*

*i)* $\{a^2\}^c = \{ x \in \mathscr{B} : ax+xa \in \{a\}^c \}$ .

*If in addition there is a* $n \in \mathscr{B}$ *with* $an = - na$, $n^2 = \mathbf{e}$ *, we get :*

*ii )* $\{a^2\}^c = \{x \in \mathscr{B} : (ax-xa)\, n \in \{a\}^c\}$ .

*iii)* $\{ x \in \mathscr{B} : ax+xa = 0 \} = \{ x \in \mathscr{B} : an \in \{a\}^c \}$ .

*iv) If moreover* $L_a$ *is injective and* $\{a\}^c$ *commutative, we have :* $\{a\}^c \cap \{n\}^c = \{a^2\}^{cc}$ .

*For* $D := C_a \mid \{a^2\}^c$ , $L := L_a \mid \{a^2\}^c$ , $R := R_a \mid \{a^2\}^c$ *we get :*

*v)* $D^k = (2L)^{k-1} D = (-2R)^{k-1} D$ *for* $k \in \mathbb{N}$.

*If* $\varphi(\zeta)$ *is an entire complex-valued function with* $\varphi(0) = 0$*, we get for* $\varphi\hat{}(\zeta) := \varphi(\zeta)/\zeta$ *( extended analytically into 0 ):*

*vi)* $\varphi(D) = \varphi\hat{}(2L)\, D = \varphi\hat{}(-2R) D$ .

*vii)* $PSp(D) \setminus \{0\} \subset 2\, PSp(L_a)$ .

**Proof:**

i) We have $(C_a)^2 = C_{a^2} - 2R_a C_a$ and for $x \in \{a^2\}^c$ : $R_a C_a(x) = -L_a C_a(x)$ .

Let $x \in \mathscr{B}$ with $ax + xa \in \{a\}^c$. Then $C_{a^2}(x) = C_a(L_a+R_a)(x) = 0$ .

Conversely let $x \in \{a^2\}^c$ : $(C_a)^2(x) = - 2R_a C_a(x) = 2L_a C_a(x)$ ,

$0 = C_a( C_a(x) - 2\, ax ) = C_a( - ax - xa )$ , $ax + xa \in \{a\}^c$ .

ii + iii) clear from $C_a(n) = 2an$ .

iv) Let $t \in \{a\}^c \cap \{n\}^c$ , $s \in \{a^2\}^c$ and $u,v \in \{a\}^c$ defined by:

$as + sa = u$ , $as - sa = vn$ .

Hence $aC_t(s) = - C_t(s)a = C_t(s)a$, i.e. $aC_t(s) = 0$ , t commutes with s, thus $\{a\}^c \cap \{n\}^c \subset \{a^2\}^{cc}$ . From $a^2 \in \{a\}^c$ and $n \in \{a^2\}^c$ we get the opposite inclusion by applying the hypothesis and Lemma 3.6.

v + vi) are easily seen by induction over k, functional calculus and Lemma 3.9.

vii) Let $r \in PSp(D)$ with $|r| > 0$ , i.e. there is $x \in \{a^2\}^c$ with $ax - xa = rx = un$ for some $u \in \{a\}^c$ .

Applying D we get : $r^2x = 2\,r\,ax$ , $rx = 2\,ax$ , $r/2 \in PSp(L_a)$ .

q.e.d.

## 3.4 Restriction of Operators

We already encountered some examples of subalgebras of the Banach algebra $\mathscr{B}$ under consideration, which are invariant under certain operators of $BL(\mathscr{B})$. The following lemma states some elementary facts about the restriction of such operators:

**Lemma 3.9:** *Let* $X$ *be a complex Banach space ,* $Y \subset X$ *a closed subspace and*

$\mathfrak{T} := \{ T \in BL(X) : T(Y) \subset Y \}$.

*i)* $\mathfrak{T} \subset BL(X)$ *is a closed subalgebra of* $BL(X)$ *with unit* **id** *, i.e.* $Sp_{BL(X)}(T) \subset Sp_{\mathfrak{T}}(T)$, $\partial Sp_{\mathfrak{T}}(T) \subset \partial Sp_{BL(X)}(T)$ , $T \in \mathfrak{T}$.

*ii)* $\mathfrak{T}|_Y \subset BL(Y)$ *and the map* $T \to T|_Y$ *ist continuous, i.e.* $Sp_{BL(Y)}(T|_Y) \subset Sp_{\mathfrak{T}}(T)$ , $T \in \mathfrak{T}$.

*iii) For* $T \in \mathfrak{T}$ *with* $Sp_{BL(X)}(T)$ *real, we have* $Sp_{\mathfrak{T}}(T)$ *and* $Sp_{BL(Y)}(T|_Y)$ *real.*

*iv) If for* $T \in \mathfrak{T}$ $Sp_{\mathfrak{T}}(T)$ *has no interior points, so doesn't* $Sp_{BL(Y)}(T|_Y)$ .

**Proof:**

i) $\mathfrak{T}$ is obviously a subalgebra of BL(X) with unit **id**.

Let $(T_n)$ be a sequence in $\mathfrak{T}$ , convergent in $BL(X)$ to $T_0$ ; for $y \in Y$ we get : $T_0(y) = \lim T_n(y) \in Y$ , because Y is closed. Hence $T_0 \in \mathfrak{T}$ , $\mathfrak{T}$ is closed in BL(X). The statement concerning the spectra is elementary , cf. [BD], p.25.

ii) For the (linear) restriction map $j : \mathfrak{T} \to L(Y)$ , $j(T) := T|_Y$ , we have

$$
\begin{aligned}
j(T) \in BL(Y) \text{ and } \|j(T)\| &= \sup \{ \|j(T)(y)\| : y \in Y \text{ with } \|y\| = 1 \} \\
&= \sup \{ \|T(y)\| : y \in Y \text{ with } \|y\| = 1 \} \\
&\leq \sup \{ \|T(x)\| : x \in X \text{ with } \|x\| = 1 \} = \|T\|.
\end{aligned}
$$

Thus $\|j\| = 1$, j is continuous. j is clearly an algebra homomorphism on $\mathfrak{T}$.

iii) follows from i .

iv) results from ii. q.e.d.

### 3.5 Pseudo-involutions

Recall that on a complex vector space X a bijective map i: $X \to X$ with:

$i(\lambda x) = \lambda^* i(x)$ , $\lambda \in \mathbb{C}$, $x \in X$ ,

$i(x+y) = i(x) + i(y)$, $x,y \in X$ ,

$i(i(x)) = x$, $x \in X$ ,

where * denotes complex conjugates, is called a *linear involution on* X .
If X is a $\mathbb{C}$-algebra and i additionally satisfies:

$i(xy) = i(y)\, i(x)$,

then it is called an *algebra involution on* X , if it satisfies

$i(xy) = i(x)i(y)$ ,

then we call it a *pseudo-involution on* X.
We denote algebra involutions by * ( $x^* := i(x)$) and pseudo-involutions by $^\sim$ , where the context has to avoid confusion.
*Self-adjoint* elements are simply elements fixed by the resp. involution.

If X is a complex Banach space with linear involution $^+$, then
$T^\sim(x) := T(x^+)^+$ for $x \in X$ and $T \in BL(X)$
defines obviously a pseudo-involution on BL(X), which is isometric if $^+$ is .

Recall that a Banach star algebra, that is a Banach algebra with involution, is called *symmetric* ([Au]), if selfadjoint elements have real spectra. We shall use this notion for pseudo-involutions too.

**Examples:**
i) If $\mathbb{C}^n$ is considered with componentwise operations and a suitable norm, then it is a commutative, symmetric, semi-simple unital Banach algebra. Here $^\sim$ means for matrices entry-wise conjugation, thus selfadjoint matrices are exactly real matrices.
Caution: With the algebra involution $i((u,v)) = (v^*,u^*)$ on $\mathbb{C}^2$, $\mathbb{C}^2$ is no longer symmetric.
ii) If X= C([0,1]) with pointwise conjugation as (isometric algebra-) involution, then a $T \in BL(X)$ is selfadjoint with respect to $^\sim$ , if and only if $C_{\mathbb{R}}[0,1]$ is invariant under T.
iii) If $X = L^2([0,1])$ then $^\sim$ as in ii makes sense as well, but is obviously different from the usual algebra involution on BL(X) .

## 4. Eigenvalues of Commutators and Inner Automorphisms

We consider an arbitrary but fixed $a \in \mathcal{B}$. Then it is well known by elementary Gelfand theory (cf. [Hi]):
$\mathrm{Sp}(C_a) \subset \mathrm{Sp}(a) - \mathrm{Sp}(a)$, with equality in the case $\mathcal{B} = M_n(\mathbb{C})$ .
$\{a\}^c \subset \{\exp(2\pi i a)\}^c$ are both closed full subalgebras of $\mathcal{B}$ with unit **e** .

**Proposition 4.1:** *If* $\mathrm{Sp}(C_a) \cap \mathbb{Z} = \{0\}$ *, then* $\exp(2\pi i a)$ *is an interior point of* $\exp(\mathcal{B})$.

**Proof:** With the entire function:

$$\theta(\zeta) = \begin{cases} 1, & \zeta = 0 \\ (\exp(\zeta)-1)/\zeta, & |\zeta| > 0 \end{cases}$$

(i.e. shortly $\theta(\zeta) = (\exp - 1)\hat{}(\zeta)$ in the sense of 3.8)
one gets the Fréchet-derivative of the exponential function at $x \in \mathcal{B}$:

$$(D \exp)_x = L_{\exp(x)} \, \theta(-C_x) \in BL(\mathcal{B}).$$

The zeroes of $\theta$ are exactly the numbers $\{ 2\pi i k : k \in \mathbb{Z}, |k| > 0 \}$ ; the spectral condition on a means therefore, that $(D \exp)_{2\pi i a} \in BL(\mathcal{B})^*$.

The assertion then follows from the Implicit Function Theorem. q.e.d.

**Proposition 4.2:** *The following statements are equivalent :*

*i)* $\{a\}^c = \{\exp(2\pi i a)\}^c$
*ii)* $\mathrm{PSp}(C_a) \cap \mathbb{Z} = \{0\}$
*iii)* $\{a\}^{cc} = \{\exp(2\pi i a)\}^{cc}$ .

*If one of these conditions is satisfied, we get :*
*iv)* *For all* $b \in \mathcal{B}$ *with* $\exp(2\pi i b) = \exp(2\pi i a)$ *:* $a \in \{b\}^{cc}$.

**Remark:** *[BD], Proposition 18.12, proves iv under the stronger condition :* $\mathrm{Sp}(C_a) \cap \mathbb{Z} = \{0\}$ .
*If* $\mathcal{B}$ *contains a non-central idempotent, then it contains nilpotent elements too.*

**Proof:**

a) iii $\Rightarrow$ i: trivial.

b) i $\Rightarrow$ iii: trivial .

c) i $\Rightarrow$ ii:

Let $y \in \mathscr{B}$ with $y \neq 0$ and $ay - ya = ky$ with $k \in \mathbb{Z}$, $|k| > 0$.

Because of

$$\exp(2\pi i C_a) = L_{\exp(2\pi ia)} R_{\exp(-2\pi ia)}$$

and $C_a{}^n(y) = k^n(y)$ for natural $n$, we have $y$ commuting with $\exp(2\pi ia)$ but not with $a$.

d) ii ⇒ i:

Suppose we had a $y \in \mathscr{B}$ commuting with $\exp(2\pi ia)$ and for which:

$$w := C_a(y) \quad , w \neq 0 .$$

Functional calculus with the function $\theta$ from 4.1 yields:

$$\theta(2i\pi C_a)(2i\pi C_a(y)) = 0.$$

Thus 0 belongs to the point spectrum of $\theta(2\pi i C_a)$. From the Fine Structure Theorem ([HP], Theorem 5.12.2, p.204) we have at least one zero of $\theta$ in the point spectrum of $2\pi i C_a$.

e) ii ⇒ iv:

Let $x \in \{b\}^c$. Then x commutes with $\exp(2\pi ia)$, hence :

$$\theta(2i\pi C_a)(2i\pi C_a(x)) = 0 .$$

As in d we see $x$ commuting with $a$. q.e.d.

If $C_a$ has no integer eigenvalues different from 0 , we have $\theta(2i\pi C_a)$ injective ! This holds for example, if $\{\exp(2\pi ia)\}^c$ is commutative, or $a \in V(\exp(2\pi ia))$ resp. $a \in \{\exp(2\pi ia)\}^{cc}$.

Let $\mathscr{C} := \{\exp(2\pi ia)\}^c$ - supposed to be non-commutative - und $D_0 \in BL(\mathscr{C})$ be the restriction of $C_a$ to $\mathscr{C}$ ; $D_0$ is a derivation from $\mathscr{C}$ into itself with $\exp(2\pi i D_0) = \mathbf{id}_{\mathscr{C}}$ (cf. 3.7) .

Hence $Sp(D_0)$ consists of a finite number of integers:

$$Sp(D_0) = \{ \lambda_0, \lambda_1, \dots , \lambda_r \} \ , \ \lambda_0 = 0 \ , \ \lambda_i \in \mathbb{Z} \text{ and}$$

$$\prod ( D_0 - \lambda_j \, \mathbf{id}_{\mathscr{C}}) = 0 .$$

Moreover there are $P_j \in BL(\mathscr{C})$, $j = 0, \dots, r$, with $P_j P_k = \delta_{jk} P_k$,

$$\mathbf{id}_{\mathscr{C}} = \Sigma \ P_j, \qquad D_0 P_j = \lambda_j P_j$$

$$\operatorname{Im} P_0 = \{a\}^c ,$$

$$D_0 = \Sigma \ \lambda_j P_j .$$

**Proposition 4.3:** $Sp_{BL(\mathscr{C})}(D_0) = PSp_{BL(\mathscr{C})}(D_0) = PSp_{BL(\mathscr{B})}(C_a) \cap \mathbb{Z}$ .

**Proof:**

i) 0 is contained in all three sets.

ii) Let $\lambda \in Sp_{BL(\mathscr{C})}(D_0)$, $|\lambda| > 0$. Then we have a $j > 0$ with $\lambda = \lambda_j$ and a $y \in \mathscr{C}$ with $z := P_j(y) \neq 0$ :

$$C_a(z) = D_0(z) = D_0 P_j(y) = \lambda_j P_j(y) = \lambda_j z .$$

Therefore $\lambda$ belongs to the set on the right and the left equality is proven .
iii) Let $\lambda \in PSp_{BL(\mathscr{B})}(C_a) \cap \mathbb{Z} \setminus \{0\}$, i.e we have a nonvanishing $y \in \mathscr{B}$ with $C_a(y) = \lambda y$. From $\lambda \in \mathbb{Z}$ we get $y \in \mathscr{C}$ , that is $\lambda \in Sp_{BL(\mathscr{C})}(D_0)$ . q.e.d.

Proposition 4.3 provides an alternative proof of 4.2 without reference to the Fine Structure Theorem.

With Lemma 3.9. and the above representation of $D_0$ one easily checks :

**Proposition 4.4:** *For* $\sigma_0 \in H$ , $\zeta_0 = \exp(\sigma_0)$ *we have :*

$$\exp(C_a(\sigma - \sigma_0))|_{\mathscr{C}} = p^*\Big( \sum_{j=0}^{r} (\zeta/\zeta_0)^{\lambda_j} P_j \Big).$$

Let *P* be the subalgebra of $BL(\mathscr{C})$ generated by $\{ P_j : j = 0, ..., r\}$ ; *P* is finite-dimensional , commutative and semi-simple but obviously reducible.

Let $\mathscr{V}$ be a closed subalgebra of $\mathscr{C}$ with $\{a\}^c \subset \mathscr{V} \subset \{\exp(2\pi i a)\}^c$ , which is invariant under *P* and $I := \{ i : P_i |_{\mathscr{V}}$ not identically $0 \}$. By assumption 0 belongs to I and:

$$Sp_{BL(\mathscr{V})}(D_0) = \{ \lambda_i : i \in I \},$$

denoting elements of *P* and their restriction to $\mathscr{V}$ by the same symbol. The case $\mathscr{V} = \{\exp(2\pi i a)\}^c$ is not excluded.

**Proposition 4.5:** *Relative to* $\mathscr{V}$ *the following statements hold :*

*i)* $\operatorname{Ker} D_0 = \operatorname{Im} P_0$, $\operatorname{Ker} P_0 = \operatorname{Im} D_0$ ,

*ii)* $\mathscr{V} = \operatorname{Ker} D_0 \oplus \operatorname{Im} D_0$ *in the sense of vector spaces ,*

*iii)* $\operatorname{Im} P_i = \{ z \in \mathscr{V} : D_0(z) = \lambda_i z\} = \{ z \in \mathscr{V} : P_i(z) = z \}$ , $\forall \; i \in I$ .

*iv)* $D_0$: $\operatorname{Im} D_0 \to \operatorname{Im} D_0$ *is bijective .*

*v)* $P_0 : \mathscr{V} \to \mathscr{V}$ *is a homomorphism of algebras if and only if* $\operatorname{Im} D_0$ *is a closed two-sided ideal in* $\mathscr{V}$ *.*

The proof is routine .

Let $\mathcal{J} := \sum \operatorname{Im} P_i$ with $i \in I^+ := \{ k \in I : \lambda_k > 0 \}$ and

$\mathcal{K} := \sum \operatorname{Im} P_i$ with $i \in I^- := \{ k \in I : \lambda_k < 0 \}$.

Then $\mathcal{J}$, $\mathcal{K}$ are closed subalgebras of $\mathcal{V}$ with $\operatorname{Im} D_0 = \mathcal{J} \oplus \mathcal{K}$; if $\mathcal{J}$, $\mathcal{K}$ are two-sided ideals in $\mathcal{V}$, then $\operatorname{Im} D_0$ is equally a closed ideal in $\mathcal{V}$. In detail :

**Proposition 4.6:** *i) The following statements are equivalent :*

*a)* $\operatorname{Im} D_0$ *is a closed two-sided ideal in* $\mathcal{V}$*.*

*b)* $\mathcal{J}\mathcal{K} \subset \mathcal{J} \oplus \mathcal{K}$, $\mathcal{K}\mathcal{J} \subset \mathcal{J} \oplus \mathcal{K}$.

*c)* $\operatorname{Im} D_0$ *is a subalgebra of* $\mathcal{V}$*.*

*ii) If* $\mathcal{J}$, $\mathcal{K}$ *are two-sided ideals in* $\mathcal{V}$*, then* $\operatorname{Im} D_0$ *is a closed two-sided ideal in* $\mathcal{V}$*. This holds especially if* $\mathcal{J} = \{0\}$ *or* $\mathcal{K} = \{0\}$.

*iii)* $\mathcal{J}$ *and* $\mathcal{K}$ *are nilpotent.*

*iv) If* $\operatorname{Sp}(D_0) \cap \operatorname{Sp}(-D_0) = \{0\}$ *, then* $\operatorname{Im} D_0$ *is a closed two-sided ideal in* $\mathcal{V}$

**Proof:**

i) $\mathcal{J}$ and $\mathcal{K}$ are obviously subalgebras of $\mathcal{V}$, which are invariant under left- or right-multiplication with elements from $\{a\}^c$. $\operatorname{Im} D_0$ is a linear subspace of $\mathcal{V}$.

a $\Leftrightarrow$ b : Easily seen from the direct sum .

a $\Rightarrow$ c: trivial.

c $\Rightarrow$ b: Let $\operatorname{Im} D_0 = \mathcal{J} \oplus \mathcal{K}$ be a subalgebra of $\mathcal{V}$.

With $j \in \mathcal{J}$, $k, k' \in \mathcal{K}$ we get : $(j+k)\, k' = jk' + kk' \in \operatorname{Im} D_0$, $kk' \in \mathcal{K} \subset \operatorname{Im} D_0$, thus $\mathcal{J}\mathcal{K} \subset \mathcal{J} \oplus \mathcal{K}$. $\mathcal{K}\mathcal{J} \subset \mathcal{J} \oplus \mathcal{K}$ follows analogously .

ii) from i.

iii) To start with, suppose $\mathcal{V} = \{ \exp(2\pi i a) \}^c$ and recall the settings of section 2.

Consider the differential equation $dv = a\, v$ with the fundamental solution $v_0(\sigma) = \exp(a(\sigma - \sigma_0))$ and its factor of automorphy $q_0 = \exp(2\pi i a)$.

It will be legitimate to anticipate Proposition 5.1; with Proposition 4.4 we get for $x \in \{q_0\}^c = \{ \exp(2\pi i a)\}^c$ :

$$p^* h_x = \exp(C_a(\sigma - \sigma_0))\,(x) = p^*\Big( \sum_{j=0}^{r} (\zeta/\zeta_0)^{\lambda_j} P_j(x)\Big).$$

That means : $\mathcal{A} = \{a\}^c \oplus \mathcal{J}$, $\Gamma_0 = P_0$, $\operatorname{Ker} \Gamma_0 = \mathcal{J}$. 5.1 says that $\mathcal{J}$ is a nilpotent two-sided ideal in $\mathcal{A}$. As each element of $\mathcal{J}$ is quasi-regular, we have $\mathcal{J} \subset \operatorname{rad}(\mathcal{A})$ .

$\mathcal{K}$ is treated analogously (using $dv = -av$ ) , yielding the nilotency of $\mathcal{K}$.

For general $\mathcal{V}$ as defined above the assertion follows from the fact, that nilpotency is inherited by subalgebras.

iv) follows from ib. q.e.d.

**Proposition 4.7:** *For the center* $Z(\mathcal{V})$ *of* $\mathcal{V}$ *we have :* $Z(\mathcal{V}) = \mathcal{V}^c \subset \{a\}^{cc}$ .

**Proof:** From : $\{a\}^c \subset \mathcal{V} \subset \{\exp(2\pi i a)\}^c$ we get by taking commutants relative to $\mathcal{B}$ :
$\{\exp(2\pi i a)\}^{cc} \subset \mathcal{V}^c \subset \{a\}^{cc}$, i.e. $\mathcal{V}^c$ is commutative, $\mathcal{V}^c \subset \mathcal{V}^{cc} \subset \{\exp(2\pi i a)\}^c$ .
So: $\{\exp(2\pi i a)\}^{cc} \subset \mathcal{V}^c \subset \{a\}^{cc} \subset \{a\}^c \subset \mathcal{V} \subset \mathcal{V}^{cc} \subset \{\exp(2\pi i a)\}^c$ .
As $Z(\mathcal{V}) := \mathcal{V} \cap \mathcal{V}^c$ , the assertion follows immediately . q.e.d.

Now let us consider a $q \in \mathcal{B}^*$ and $\alpha := \alpha_q \in BL(\mathcal{B})$ , the inner automorphism of $\mathcal{B}$ defined by q . Again, it is well known (cf. [BD]) : $Sp(\alpha) \subset \{ \lambda\mu^{-1} : \lambda,\mu \in Sp(q)\}$ and equality holds for matrix-algebras $M_n(\mathbb{C})$ .

If $\mu \in PSp(\alpha)$ , so pick a non-zero eigenvector $z \in \mathcal{B}$ with $\alpha(z) = \mu z$.
For $k \in \mathbb{N}$ we have $\alpha(z^k) = \mu^k z^k$ . Thus $|\mu| = 1$ or all eigenvectors for $\mu$ are nilpotent.

Let $k > 1$ be a natural number; then : $\{q\}^c \subset \{q^k\}^c$ , hence $\beta := \alpha \,|\, \{q^k\}^c$ is an inner automorphism $\beta\colon \{q^k\}^c \to \{q^k\}^c$ with $\beta^k = \mathbf{id} \,|\, \{q^k\}^c$ .
Let $\mathfrak{W}_k := \{ \lambda \in \mathbb{C}: \lambda^k = 1 \}$ denote the set of the kth roots of unity ; we see $Sp(\beta) \subset \mathfrak{W}_k$ .
Analogously to the arguments used above, it follows that $\beta$ possesses a pure point spectrum:
$Sp(\beta) = \{ \lambda_1=1, \lambda_2, ..., \lambda_n\} \subset S^1$ , $n \le k$, $\lambda_m \in \mathfrak{W}_k$ , $m= 1,...,n$ .
$\{q^k\}^c = \{q\}^c \oplus \mathcal{J}$ in the sense of vector spaces, where $\mathcal{J}$ is the subspace spanned by the eigenvectors corresponding to eigenvalues of $\beta$ different from 1 .

As $Sp(\beta) = PSp(\beta) = PSp(\alpha) \cap \mathfrak{W}_k$ ,
it follows: $\{q\}^c = \{q^k\}^c$ if and only if $PSp(\alpha) \cap \mathfrak{W}_k = \{1\}$ .
With $\cup\, \mathfrak{W}_k = e^{2\pi i \mathbb{Q}}$ we complete the proof of :

**Proposition 4.8:** *We have* $\{q\}^c = \{q^k\}^c$ *for all* $k \in \mathbb{N}$ *if and only if* $PSp(\alpha) \cap e^{2\pi i \mathbb{Q}} = \{1\}$ .

**Corollary 4.9:** *If* $Sp(q)$ *is real and purely negative or purely positive ,  then* $\{q\}^c = \{q^k\}^c$ *for all* $k \in \mathbb{N}$ .

**Proposition 4.10:**

*If $\mathscr{B}$ contains no divisors of zero, then* $PSp(\alpha)$ *is a semi-group with unit 1 in* $S^1$. *If moreover* $q \in \exp(\mathscr{B})$, *then* $PSp(\alpha) = \{1\}$.

**Proof:**

a) If $\mathscr{B}$ contains no divisors of zero, then $\alpha$ has no eigenvalues of modulus $> 1$, because the corresponding eigenvectors were nilpotent.

Let $\lambda,\mu \in PSp(\alpha)$ and x,y corresponding eigenvectors, then the product xy is non-zero, thus an eigenvector corresponding to the eigenvalue $\lambda\mu$.

b) If $q = \exp(y)$, we have $\alpha = \exp(C_y)$ and $PSp(\alpha) = \exp(PSp(C_y))$ due to the Fine Structure Theorem. But $PSp(C_y) = \{0\}$, because eigenvectors to non-zero eigenvalues of a commutator were nilpotent. q.e.d.

**Proposition 4.11:** *Let $\mathscr{X}$ be a commutative semi-simple Banach algebra with unit* **e** *and* $\mathscr{B} = BL(\mathscr{X})$.

*If* $u : \mathscr{X} \to \mathscr{X}$ *is a bijective algebra homomorphism, then* $u, u^{-1} \in BL(\mathscr{X})$ *and :*

*i)* $PSp(u) \subset PSp(\alpha_u) \cap S^1$.

*ii) If* $\lambda \in PSp(\alpha_u) \cap S^1$ *, then* $\lambda \in PSp(u)$ *if and only if there is a* $T \in BL(\mathscr{X})$ *with* $\alpha_u(T) = \lambda T$ *and* T(**e**) *different from 0 .*

*iii)* $\lambda \in PSp(u) \Rightarrow \lambda^k \in PSp(u)$ *for all* $k \in \mathbb{N}$.

*iv) If $\mathscr{X}$ is symmetric , then iii) holds for all* $k \in \mathbb{Z}$.

*If $\mathscr{X}$ is even a B*-algebra, then* $PSp(\alpha_u) \subset Sp(\alpha_u) \subset S^1$.

**Proof:**

Algebra homomorphisms from a Banach algebra into a commutative semi-simple Banach algebra are always continuous ([BD], Theorem 17.8, p. 83) .

a) Let $f \in \mathscr{X}$ different from 0 and $\lambda \in \mathbb{C}$ with $u(f) = \lambda f$.

Since u is an algebra homomorphism, we have $\alpha_u(L_f) = \lambda L_f$; since $\mathscr{X}$ is semi-simple $L_f$ cannot be nilpotent. Thus $|\lambda| = 1$ and $\lambda^k \in PSp(u)$ for all $k \in \mathbb{N}$ .

i, ii and iii are thus obvious.

b) If $\mathscr{X}$ is symmetric, u is a *-homomorphism, thus $u(f^*) = u(f)^* = \lambda^* f^*$. In the case a of a B* - algebra all automorphisms are isometries. q.e.d.

Though without any direct relation to the main theme of this paper let us state as application of 4.1 and 4.2:

**Proposition 4.12:** *Let $\mathcal{B}$ be a complex Banach algebra with unit* $\mathbf{e}$ *which contains no quasi-nilpotent elements.*
*Let* $\mathfrak{P}$ *be the set of all idempotents of* $\mathcal{B}$ *(including* 0 *and* $\mathbf{e}$) *and* $\mathfrak{G} := \{ x \in \mathcal{B} : x^2 = \mathbf{e}\}$.
*Then* $\mathfrak{P}$ , $\mathfrak{G}$ *are contained in the center* $\mathcal{B}^c$ *of* $\mathcal{B}$ . $\mathfrak{G} \subset \exp(\mathcal{B})$ *is a closed, discrete, commutative normal subgroup of* $\mathcal{B}^*$ . *The map* $j(p) := 2p - \mathbf{e}$, $p \in \mathfrak{P}$, *defines a homeomorphism* $j : \mathfrak{P} \rightarrow \mathfrak{G}$ , *hence* $\mathfrak{P}$ *is a closed, discrete commutative subset of* $\mathcal{B}$ .
*Every finite dimensional subalgebra of* $\mathcal{B}$ *is contained in* $\mathcal{B}^c$.
*For each* $p \in \mathfrak{P}$ $p\mathcal{B}$ *is a closed, two-sided ideal of* $\mathcal{B}$.

The easy **proof** is sketched as follows: for $p \in \mathfrak{P}$ we get from 4.2 : $p \in \mathcal{B}^c$ because $C_p$ has no non-trivial eigenvalues. But then $C_p = 0$ and (the proof of) 4.1 shows that the points of $\mathfrak{P}$ are discrete in $\mathcal{B}$ . An element of a finite dimensional subalgebra has finite spectrum and by the absence of quasi-nilpotents is a complex linear combination of some idempotents. The rest is straightforward.

**Corollary 4.13:** *A finite dimensional Banach algebra without nilpotent elements is commutative.*

This corollary is due to Hirschfeld and Zelazko resp. Nemirowskii. It may be derived directly from 4.2. For an alternative proof cf. [Au], p.44.

**Corollary 4.14:** *If* $\mathcal{B} = BL(H)$ *for some complex Hilbert space* $H$ *and* $M \subset \mathcal{B}$ *is a selfadjoint subset such that* $M^c$ *is not commutative, then* $M^c$ *contains nilpotent elements.*

**Proof:**
If $M \subset \mathcal{B}$ is a selfadjoint subset ( i.e. $M$ is invariant under *, the usual algebra involution on $\mathcal{B}$) , then $M^c = (M^c)^{cc}$ is invariant under * too, hence $M^c$ is a von-Neumann-algebra ( = weakly closed = strongly closed) . The idempotents in $M^c$ span a norm-dense subspace (see [Ne] ). If there were no nilpotent elements in $M^c$ then we had there a norm-dense commutative subset. q.e.d.

## 5. The Differential Equation of the Transformation of Monodromy

We now return to the situation described in section 2.

From the definitions it follows immediately, that there is an uniquely determined holomorphic $\mathscr{B}^*_0$-valued function $g_0$ on E* with $p^*g_0 = v_0 q_0 v_0^{-1}$ .

One easily checks that $g_0$ is a solution of the differential equation ( $\zeta \in E^*$):

$$(**) \qquad \zeta g'(\zeta) = C_{f(\zeta)}(g(\zeta))$$

with $g_0(\zeta_0) = q_0$ ( $\zeta_0 = \exp(\sigma_0)$).

$g_0$ is an invariant of the differential equation (*); if we started with an arbitrary ( not necessarily normalized ) fundamental solution v on H with factor of automorphy q , then we would get $v\, q\, v^{-1} = v_0\, q_0\, v_0^{-1} = p^*g_0$ .

Thus the range of $g_0$ outside 0, $g_0(E^*) \subset \mathscr{B}^*_0$ , contains exactly the factors of automorphy corresponding to normalized fundamental solutions of (*) .
Due to its particular character we call $g_0$ the *transformation of monodromy of (*) at 0* following Deligne in [De], p. 53, although his point of view is quite different.

**Remark:** In connection with Proposition 2.1 we observe, that $g_0$ is k-invariant if f is . If f is k-invariant and $PSp(C_{f(0)}) \cap \mathbb{N} = \varnothing$ , then all solutions of (**), which are single-valued and holomorphic in E , are k-invariant.

(Here only ) to compactify notation, we define for an open set $U \subset \mathbb{C}$ :
$\mathscr{O}^{\mathscr{B}}(U) := \{ h : U \to \mathscr{B} \text{ holomorphic}\}$,
$\mathscr{O}^{\mathscr{B}^*}(U) := \{ h : U \to \mathscr{B}^* \text{ holomorphic}\}$,
$\mathscr{M}^{\mathscr{B}}(U) := \{ h : U \to \mathscr{B} \text{ meromorphic}\}$,
(where as usual *meromophic* means holomorphic in $U \setminus D$ , D a discrete, closed set; at the isolated points of D a meromorphic function has at most poles ( of finite order )) .

$\mathcal{M}^{\mathcal{B}*}(U) := \{ h \in \mathcal{M}^{\mathcal{B}}(U)$ : outside a discrete, closed set h is holomorphic and the values of h are regular; the function obtained by pointwise taking inverses again is meromorphic, shortly $h^{-1} \in \mathcal{M}^{\mathcal{B}}(U) \}$.

( For more details concerning these additive resp. multiplicative pre-sheaves we refer to [Al].)

$\mathfrak{L} := \{ h \in \mathcal{O}^{\mathcal{B}}(E^*) \ : \ \zeta h'(\zeta) = C_{f(\zeta)}(h(\zeta))\ ,\ \zeta \in E^* \}$,

$\mathfrak{h} := \mathfrak{L} \cap \mathcal{O}^{\mathcal{B}}(E)$ .

Thus $\mathfrak{L}$ is the set of the solutions of (**) , which are defined and single-valued in E* , $\mathfrak{h} \subset \mathfrak{L}$ is the subset of solutions which can be extended analytically into 0 .

**Definition and Proposition 5.1:**

*a)* $\mathfrak{L} \subset \mathcal{M}^{\mathcal{B}}(E) \cap \mathcal{O}^{\mathcal{B}}(E^*)$ , $\mathfrak{h} \subset \mathcal{O}^{\mathcal{B}}(E)$ *are unital* $\mathbb{C}$*-algebras with operations understood pointwise ;* $\mathfrak{h}$ *is a subalgebra of* $\mathfrak{L}$*. Common unit is the function constantly equal to* **e** .

*b) There is a* $M \geq 0$*, so that each* $h \in \mathfrak{L}$ *has in 0 a pole or a zero of order at most* M .

*c) For* $h \in \mathcal{M}^{\mathcal{B}}(E)$ *we have:* $h \in \mathfrak{L} \Leftrightarrow \exists\ x \in \{q_0\}^c :\ p^*h = v_0\, x\, v_0^{-1}$ *on* H.

*d) The map* $\Phi : \mathfrak{L} \to \{q_0\}^c$ *, defined by* $\Phi(h) := h(\zeta_0)$*,* $h \in \mathfrak{L}$ *, is a bijective algebra homomorphism.*

*( For* $x \in \{q_0\}^c$ *we occasionally write* $h_x := \Phi^{-1}(x)$*.)*

*e) The algebra* $\mathcal{A} := \Phi(\mathfrak{h})$ *is a closed subalgebra of* $\{q_0\}^c$ , $\mathbf{e} \in \mathcal{A}$*.*

*f) The map* $\Gamma_0 : \mathcal{A} \to \mathcal{B}$ *, defined by* $\Gamma_0(x) := \Phi^{-1}(x)(0)$ , $x \in \mathcal{A}$ *, is a continuous algebra homomorphism* $\Gamma_0 : \mathcal{A} \to \{a_0\}^c$ .

*g) Any* $x \in \mathrm{Ker}\ \Gamma_0$ *,* $x \neq 0$*, is nilpotent.*

*h) If* $\mathrm{PSp}(C_{f(0)}) \cap -\mathbb{N} = \varnothing$ *, then* $\mathcal{A} = \{q_0\}^c$.

*i) If* $\mathrm{PSp}(C_{f(0)}) \cap +\mathbb{N} = \varnothing$ *, then* $\Gamma_0$ *is injective .*

**Proof:**

**a,b,c,d,f**: Lemma 3.1 (including the remark) says that $\{v_0 x v_0^{-1} : x \in \mathcal{B}\}$ is the set of all solutions of (**) lifted to H; but such a solution is invariant under Deck(H/E*) if and only if x commutes with $q_0$.

One easily checks that $\mathfrak{L}$ is a $\mathbb{C}$-algebra and holomorphy in 0 is preserved under pointwise addition and multiplication.

The map $\Psi : \{q_0\}^c \to \mathfrak{L}$, $\Psi(x) := h_x$ , where $p^*h_x = v_0 x v_0^{-1}$, is an algebra homomorphism and bijective by Lemma 3.1.

That h $\in \mathcal{L}$ can be extended meromorphically into 0 - with globally bounded order - is an immediate consequence of Lemma 3.4 .
That the range of $\Gamma_0$ belongs to $\{f(0)\}^c$ , is easily seen by the Taylor series expansion of a solution h $\in \mathfrak{H}$.

**e**: If for $k \in \mathbb{Z}$ , $0 \le k$ , we set :

$$\Gamma_k(x) := 1/(2\pi i) \int_C v_0(\sigma)xv_0(\sigma)^{-1}(\exp(+k\sigma))d\sigma$$

with $\sigma \in H$ and C the straight line joining $\sigma_0$ and $\sigma_0+2\pi i$ ,
we get a sequence of well-defined linear and bounded maps: $\Gamma_k : \{q_0\}^c \to \mathcal{B}$ .
As already observed, only finitely many $\Gamma_k$ are different from zero and we have:

$$\mathcal{A} = \bigcap_{k>0} \text{Ker } \Gamma_k \ .$$

Thus $\mathcal{A}$ is closed.

**h,i**: clear with appropriate Taylor series for functions from $\mathcal{L}$ resp. $\mathfrak{H}$ .
**g:** Let $h_x \in \mathfrak{H}$ with $h_x(0) = 0$. If $h_x$ is not identically $= 0$, we have an integer $k > 0$ with $h_x(\zeta) = \zeta^k g(\zeta)$, g holomorphic in E, c := g(0) , $c \neq 0$.
For any natural m $h_x{}^m$ is a solution of (**) too. If all these powers of $h_x$ were not identically zero, their leading Taylor-coefficient at 0 would provide an eigenvector of $C_{f(0)}$ corresponding to an integer eigenvalue $k_m$, $k_m \geq km$.
But the point spectrum of $C_{f(0)}$ is clearly bounded, $h_x$ is nilpotent in $\mathfrak{H}$ and so is x. q.e.d.

For the transformation of monodromy we get especially ( remember $a_0 = f(0)$ ) :

**Proposition 5.2:**
*i) If for the solution* $g_0$ *of (**) we set*

$$g_0(0) := \exp(2\pi i a_0) \ ,$$

*then* $g_0$ *is holomorphic in* E, *i.e.* $q_0$ , $q_0^{-1} \in \mathcal{A}$ *and* $\Gamma_0(q_0) = \exp(2\pi i a_0)$ .
*ii) If (*) admits of a Frobenius solution then*

$$\exp(2\pi i a_0) = \lim \exp(\sigma C_{f(0)})(q_0) \ \textit{for} \ \sigma \in H \ \textit{real}, \sigma \to -\infty \ .$$

**Proof:** i) For $s_0$, s $\in$ H real with $s<s_0$ and t real we consider the fundamental solution $v_{s,t}$ of (*) with: $v_{s,t}(s+2\pi it) = \mathbf{e}$.
Then we get respectively a unique $q_{s,t} \in \mathcal{B}$ with

$v_{s,t} = v_{s,0}.q_{s,t}$, that is

$q_{s,t} = v_{s,0}{}^{-1}(s+2\pi it)$,

$q_{s,0} = \mathbf{e}$.

On the other hand we have:

$v_{s,t} = v_0.v_0(s+2\pi it)^{-1}$,

$\mathfrak{g}_0{}^*v_{s,0} = v_{s,0}.v_0(s)q_0v_0(s)^{-1}$, i.e.

$q_{s,1} = v_{s,0}{}^{-1}(s+2\pi i) = v_0(s)q_0{}^{-1}v_0(s)^{-1}$.

Fix s and consider q as function of t alone, then q is differentiable in the real sense and we get:

$q'(t) = -q(t)2\pi if(e^se^{2\pi it})$, $q(0) = \mathbf{e}$,

a differential equation, which for $s \to -\infty$ simplifies to

$u'(t) = -u(t)2\pi ia_0$, $u(0) = e$,

with the solution $u(t) = \exp(-2\pi ia_0t)$.

Holomorphy of f and Lemma 3.2 provide the existence of:

$$\lim_{s\to-\infty} v_0(s)q_0{}^{-1}v_0(s)^{-1} = \lim_{s\to-\infty} q_{s,1} = \exp(-2\pi ia_0) :$$

Let $\varepsilon > 0$ and $|s|$ large enough, so that $\| f(e^se^{2\pi it})\| < \| a_0 \| + \varepsilon$ .

Lemma 3.2 yields for q(t):

$\|q(t)\| \leq \exp(2\pi(\| a_0 \| + \varepsilon)t)$ for $t \in [0,1]$.

This implies for $r(t) := q(t) - u(t)$ :

$\|r'(t)\| \leq 2\pi\|r(t)\| \| a_0 \| + \exp(2\pi(\| a_0 \| + \varepsilon)t)\exp(s) M$

$\leq 2\pi\|r(t)\| \| a_0 \| + \exp(2\pi(\| a_0 \| + \varepsilon))\exp(s) M$,

where $M > 0$ be chosen so that $\|(f(\zeta) - a_0)/\zeta\| \leq M$, $\zeta = e^se^{2\pi it}$ and s as above.

Application of Lemma 3.2 again yields:

$\|r(t)\| \leq \exp(2\pi(\| a_0 \| + \varepsilon))\exp(s) M \exp(2\pi\| a_0 \| t)$,

and here the right hand side tends to 0 ( especially for t = 1 ) as $s \to -\infty$ .

If $g_0{}^{-1}$ was not bounded near 0, 5.1 would provide a $k > 0$ and a $h_0 \in \mathcal{O}^{\mathcal{B}}(E)$ with

$h_0(0) =: c \neq 0$, $g_0{}^{-1}(\zeta) = \zeta^{-k}h_0(\zeta)$, such that

$p^*g_0{}^{-1}(s) = v_0(s)q_0{}^{-1}v_0(s)^{-1} = e^{-ks}h_0(e^s)$ .

As $h_0$ is continuous in 0, this would mean c = 0: let $s \to -\infty$.

Hence $g_0{}^{-1}$ is bounded in E , consequently holomorphic near 0 with

$g_0{}^{-1}(0) = \exp(-2\pi ia_0)$ .

Taking inverses yields the assertion.

ii) Clear from the definition of a Frobenius solution and i.

q.e.d.

This Proposition especially shows, that there are fundamental solutions of (*) with a factor of automorphy arbitrarily close to $\exp(2\pi i a_0)$ . $q_0$ can always be written as product of at most two factors from $\exp(\mathcal{B})$ , $g_0(E^*) \subset \exp(\mathcal{B})^2$ .
Therefore it is highly questionnable whether each $q \in \mathcal{B}^*_0$ can figure as factor of automorphy of a fundamental solution of a differential equation (*) . In [Al] it is shown, that this is only the case if no restrictions on the behavior of f in 0 are imposed.

If $\exp(\mathcal{B})$ is open, we see $q_0 \in \exp(\mathcal{B})$. This completes the proof of Theorem 1.1.

Proposition 5.2 - though without reference to $g_0$ - is essentially proved for the case $\mathcal{B} = M_n(\mathbb{C})$ for example in [De] and [Ga]; a simple example in $M_2(\mathbb{C})$ is used ( cf. [Ga], Vol.2, p.145f) to show, that $q_0$ and $\exp(2\pi i a_0)$ in general are not conjugate to each other - contrary to a former assertion of Volterra.

In the case $\mathcal{B} = M_n(\mathbb{C})$ we get from 5.1 and 5.2, that $q_0$ and $\exp(2\pi i a_0)$ have the same characteristic polynomial and consequently the the same eigenvalues.
For general $\mathcal{B}$ this has to be refined:

**Proposition 5.3:** *For all* $x \in \mathcal{A}$ *we have*

$$\mathrm{Sp}_{\mathcal{A}}(x) = \mathrm{Sp}_{\mathcal{B}}(\Gamma_0(x)) .$$

**Proof:**
Let $x \in \mathcal{A}$ and $h_x$ the corresponding solution of (**); $h_x$ is holomorphic in 0 and with $c:=h_x(0)$ we have $c = \Gamma_0(x)$. For $\zeta \in E^*$ we get from 5.1c $\mathrm{Sp}_{\mathcal{B}}(h_x(\zeta)) = \mathrm{Sp}_{\mathcal{B}}(x)$.
Since $\Gamma_0$ is an algebra homomorphism: $\mathrm{Sp}_{\mathcal{B}}(\Gamma_0(x)) \subset \mathrm{Sp}_{\mathcal{A}}(x)$ .
With the continuity of $h_x$ at 0 and the upper semicontinuity of the spectrum we see :
$\mathrm{Sp}_{\mathcal{B}}(x) \subset \mathrm{Sp}_{\mathcal{B}}(\Gamma_0(x))$ , combining:
$\mathrm{Sp}_{\mathcal{B}}(x) \subset \mathrm{Sp}_{\mathcal{B}}(\Gamma_0(x)) \subset \mathrm{Sp}_{\mathcal{A}}(x)$ .

The polynomial-convex hulls of $\mathrm{Sp}_{\mathcal{B}}(x)$ and $\mathrm{Sp}_{\mathcal{A}}(x)$ in $\mathbb{C}$ are equal ( one has to fill the holes, that are the bounded components of the resp. resolvent set , see [Bou]).

Suppose now, we had a $\lambda \in \mathrm{Sp}_{\mathcal{A}}(x) \setminus \mathrm{Sp}_{\mathcal{B}}(\Gamma_0(x))$ . Consequently $\lambda$ lies in a hole of $\mathrm{Sp}_{\mathcal{B}}(\Gamma_0(x))$ , thus in a hole of $\mathrm{Sp}_{\mathcal{B}}(x)$, too .
Hence $\lambda\mathbf{e} - c$ is regular in $\mathcal{B}$; now $(\lambda\mathbf{e} - h_x)^{-1}$ is a solution of (**) continuous in 0 (because of $(\lambda\,\mathbf{e} - h_x)^{-1} = (\lambda\mathbf{e} - c)^{-1}\ (\mathbf{e} - (h_x - c)(\lambda\mathbf{e} - c)^{-1})^{-1}$ ) ). That means $(\lambda\,\mathbf{e} - x)^{-1} \in \mathcal{A}$, $\lambda$ doesn't belong to $\mathrm{Sp}_{\mathcal{A}}(x)$. Contradiction! q.e.d.

**Corollary 5.4:** *i)* $\Gamma_0(\mathcal{A})$ *is a full subalgebra of* $\{a_0\}^c$ .

*ii) The smallest full subalgebra of* $\{a_0\}^c$ *containing* $\exp(2\pi i a_0)$ *lies in* $\Gamma_0(\mathcal{A})$ .

*iii) If* $\mathcal{A}$ *is a full subalgebra of* $\{q_0\}^c$, *then* $\mathrm{Sp}_{\mathcal{B}}(x) = \mathrm{Sp}_{\mathcal{B}}(\Gamma_0(x))$ *for all* $x \in \mathcal{A}$.

*iv) We have* $\sigma(x) = \sigma(\Gamma_0(x))$ *for all* $x \in \mathcal{A}$, *denoting by* $\sigma(x)$ *for* $x \in \mathcal{B}$ *the full spectrum of* $x$ *( that is the polynomial-convex hull of* $\mathrm{Sp}(x)$ *, cf. [Au] ).*

*v) If* $\sigma(x)^\circ = \emptyset$ *for all* $x \in \mathcal{B}$, *then* $\mathcal{A}$ *is a full subalgebra of* $\mathcal{B}$.

*If* $\mathcal{B}$ *contains no divisors of zero or if* $a_0 \in \mathcal{B}^c$ *, we have :*

$$\mathcal{A} = \{q_0\}^c \text{ and } \Gamma_0 : \mathcal{A} \to \{a_0\}^c \text{ is an injective algebra homomorphism.}$$

*In all three cases* $\mathrm{Sp}_{\mathcal{B}}(x) = \mathrm{Sp}_{\mathcal{B}}(\Gamma_0(x))$ *for all* $x \in \mathcal{A}$.

*vi) If* $\exp(2\pi i a_0) = \mathbf{e}$, *then* $q_0 = \mathbf{e} + n, n \in \mathcal{B}$ *with* $n = 0$ *or* $n$ *nilpotent.*

*vii) From* $q_0 = \mathbf{e}$ *follows* $\exp(2\pi i a_0) = \mathbf{e}.$

It should be noticed, that no statements concerning the closedness of $\Gamma_0(\mathcal{A})$ are possible up to now.

**Proposition 5.5:** *If* $X$ *is a complex Banach space and* $\mathcal{B} = \mathrm{BL}(X)$, *then* $\mathrm{PSp}(q_0) \subset \mathrm{PSp}(\exp(2\pi i a_0))$ .

**Proof:**

Take an eigenvalue r of $q_0$ and a non-zero eigenvector $x \in X$ with $q_0(x) = rx$.

r is different from 0 , i.e. we have a complex number s with $r = \exp(2\pi i s)$.

$u(\sigma) := v_0(\sigma)(x)$ is holomorphic in H with values in X . From $(\mathfrak{g}_0)^* u = r\,u$ we see that $v(\sigma) := u(\sigma)\exp(-s(\sigma-\sigma_0))$ is invariant under $(\mathfrak{g}_0)^*$: there is a holomorphic X-valued function h on $E^*$ with $v = p^* h$ . As in Lemma 3.4 we can argue, that h has in 0 at most pole of finite order.

On the other hand we have

$p^* g_0(\sigma)(u(\sigma)) = r u(\sigma)$ , i.e. $g_0(\zeta)(h(\zeta)) = r\,h(\zeta)$ .

Therefore the leading coefficient in the Laurent-series of h at 0 is an eigenvector of $\exp(2\pi i a_0)$ corresponding to the eigenvalue r. q.e.d.

Analogously to $\mathfrak{f} := \{f(\zeta) : \zeta \in E\}$ we set $\mathfrak{W} := \{v_0(\zeta) : \zeta \in H\}$.

From Proposition 5.1 one immediately deduces $\mathfrak{f}^c = \mathfrak{W}^c$. We thus get:

**Proposition 5.6:**

*a)* $\mathfrak{f}^c \subset \{a_0\}^c \cap \{q_0\}^c$ .

*b) If* $\{a_0\}^c$ *is commutative, then :* $\mathfrak{f}^c \subset \{a_0\}^c \subset \mathfrak{f}^{cc}$.

*c) If* $\{a_0\}^c$ *is commutative and if* $\mathrm{PSp}(C_{f(0)}) \cap \mathbb{Z} = \{0\}$ *, then* $\{q_0\}^c$ *is commutative too and*

$$\mathfrak{f}^c \subset \{q_0\}^c \subset \mathfrak{f}^{cc}.$$

Generally we assumed $\mathfrak{F}$ to be non commutative, the inclusion on the right hand side in c is strict.
If we had $q_0 \in \mathfrak{F}^c$, we had likewise $q_0 = \exp(2\pi i a_0)$ . Because under the assumptions of c we have $\mathcal{A} = \{q_0\}^c$ and $\Gamma_0 |_{\mathfrak{F}^c} = \mathbf{id}$ .
If now $q_0 \in \mathfrak{F}^c$ it would follow $\mathfrak{F}^{cc} = \{q_0\}^c$ , i.e. $\mathfrak{F}$ is commutative. Thus also the inclusion on the left of c is strict.
Moreover we see, that under the assumptions of c the function $g_0$ is not constant (cf. example 6.16).

**Example 5.7:**
Take $a,b \in \mathcal{B}$ , two non-commuting elements, and consider the hypergeometric differential equation ( [Hi]):

$$\zeta w'(\zeta) = ( a + b\, \zeta/(1-\zeta) )\, w(\zeta) ,\ \zeta \in E^* .$$

Here we have $\mathfrak{F}^c = \{a\}^c \cap \{b\}^c$ .
This holds of course also for the equation:

$$\zeta w'(\zeta) = ( a + bh( \zeta) )\, w(\zeta) ,\ \zeta \in E^* ,$$

where h is an arbitrary holomorphic complex-valued function on E with $h(0) = 0$.

Let us consider in particular the case $f(\zeta) = a + b\, \zeta$ , $f'(\zeta) = b$. If $g_0$ is constant, then necessarily b and $\exp(2\pi i a)$ would commute. If additionally $\mathrm{PSp}( C_a ) \cap \mathbb{Z} = \{0\}$ , then $g_0$ is not constant, independently of the commutativity of $\{a\}^c$ .
If we take $f(\zeta) = a + b\, \zeta$ and consider the corresponding differential equation at the singular point $\infty$ , then we get a pole of second order. In this case $\Phi^{-1}(\mathcal{A})$ consists of constant functions only which commute with b and a .

Whether or not $q_0$ has a logarithm in $\mathcal{B}$ , we have the following immediate consequences of propositions 5.1 and 4.2:

**Proposition 5.8:** *Suppose* $\{ a_0 \}^c$ *to be commutative and* $\mathrm{PSp}( C_{f(0)} ) \cap \mathbb{Z} = \{0\}$ . *Then* $\{q_0\}^c$ *is commutative too and :*
$\{ a_0 \}^c = \Gamma_0 (\mathcal{A})^c = \Gamma_0 (\mathcal{A})^{cc} , \{ q_0 \}^c = \mathcal{A} = \mathcal{A}^c = \mathcal{A}^{cc}$ .

**Proposition 5.9:** *Suppose* $\{a_0\}^c$ *to be commutative and* $\mathrm{PSp}( C_{f(0)} ) \cap \mathbb{Q} = \{0\}$ , *then* $\{q_0\}^c$ *is commutative too and :*

*i)* $\{q_0\}^c = \{q_0{}^k\}^c$ *for all* $k \in \mathbb{Z} \setminus \{0\}$.

*ii) We have* $q_0 \in \exp(\mathcal{B})$ *if and only if there is a* $k \in \mathbb{Z} \setminus \{0\}$ *with* $q_0{}^k \in \exp(\mathcal{B})$ .

**Proof:**

i) It is sufficient to consider $k > 0$.
With $f(\zeta)$ we have $f_k(\zeta) := kf(\zeta^k)$ holomorphic on $E$ with $f_k(0) = ka_0$ .
If $\sigma_k := \sigma_0/k \in H$ , then $v_k(\sigma) := v_0(k\sigma)$ is the fundamental solution of $dv = p^*(f_k d\zeta/\zeta)v$ on $H$ with $v_k(\sigma_k) = \mathbf{e}$.
Since $(\mathfrak{g}_0)^* v_k = v_k q_0{}^k$ and $PSp(\, C_{kf(0)}\,) \cap \mathbb{Z} = \{0\}$ we get from propositions 5.1 and 6.6 injective, continuous homomorphisms: $P_k : \{q_0{}^k\}^c \rightarrow \{ka_0\}^c = \{a_0\}^c$ with
$P_k\,(q_0{}^k) = \exp(2\pi k i a_0)$ , $P_1 = \Gamma_0$.
Especially $\{q_0{}^k\}^c$ is commutative and since $\{q_0\}^c \subset \{q_0{}^k\}^c$ we get from Lemma 3.6:
$\{q_0\}^c = \{q_0{}^k\}^c$ .
ii) Let $k > 1$ with $q_0{}^k \in \exp(\mathcal{B})$ , i.e. there is a $y \in \mathcal{B}$ with $q_0{}^k = \exp(y)$ .
Then $y \in \{q_0{}^k\}^c = \{q_0\}^c$; consequently $q_0{}^k \in (\{q_0\}^c)^*{}_0$ and thus $q_0 \in (\{q_0\}^c)^*{}_0 = \exp(\{q_0\}^c)$, because the quotient group $(\{q_0\}^c)^* / (\{q_0\}^c)^*{}_0$ contains no elements of finite order(cf. [HP]) .
q.e.d.

**Remark 5.10:**
As a result of Proposition 5.5 we have always $PSp({}^{\alpha}q_0) \subset PSp(\exp(2\pi i C_{f(0)})$ , since ${}^{\alpha}g_0$ satisfies a differential equation of the form (**) . By the condition on $PSp(\, C_{f(0)}\,)$, the Fine Structure Theorem and Proposition 4.8 we get $\{q_0\}^c = \{q_0{}^k\}^c$ independently of the commutativity of $\{a_0\}^c$ . But the latter enters essentially into assertion ii.

As briefly as possibly we state an application to algebras with pseudo-involutions (see section 3 ). Suppose we had a continuous pseudo-involution $^{\sim}$ on $\mathcal{B}$ .
Let $U \subset \mathbb{C}$ be open with $U^* = U$. If for $g \in \mathcal{O}^{\mathcal{B}}(U)$ we set $g^x(\zeta) = g(\zeta^*)^{\sim}$ , $\zeta \in U$, we get a pseudo-involution $^x : \mathcal{O}^{\mathcal{B}}(U) \rightarrow \mathcal{O}^{\mathcal{B}}(U)$ , because by continuity of $^{\sim}$ $^x$ commutes with complex differentiation.
We suppose $f^x(\zeta) = f(\zeta)$ on $E$ , that is $a_k{}^{\sim} = a_k$ , $k = 0,1,2,....$
We note especially that $\{a_0\}^c$ is closed under $^{\sim}$ .

Since $(p^*f)^x(\sigma) = (p^*f)(\sigma^*)^{\sim} = f(e^{\sigma^*})^{\sim} = (p^*(f^x))(\sigma)$ for $\sigma \in H$ we get
$(v_0{}^x)'(\sigma) = (v_0{}')^x(\sigma) = v_0{}'(\sigma^*)^{\sim} = f(e^{\sigma^*})^{\sim}\, v_0(\sigma^*)^{\sim} = f(e^{\sigma})\,(v_0{}^x)(\sigma)$ .

If - without loss of generality - we suppose $\sigma_0$ to be real, we get $v_0 = v_0{}^x$ ,
hence: $q_0{}^{\sim} = q_0{}^{-1}$ . $\{q_0\}^c$ is closed under $^{\sim}$ , and we have $\Gamma_0(x^{\sim}) = \Gamma_0(x)^{\sim}$ on $\mathcal{A} = \mathcal{A}^{\sim}$.
From $v_0 = v_0{}^x$ we get $g_0{}^x = g_0{}^{-1}$ .

If $\{a_0\}^c$ is symmetric with repect to $^\sim$ , then by Proposition 5.3 $\mathscr{A}$ is symmetric too.

It is then straightforward to complete the proof of:

**Proposition 5.11:** *Let* $\mathscr{B}$ *be equipped with a continuous pseudo-involution* $^\sim$ *and* f *satisfy* $f^{x}(\zeta) = f(\zeta)$ *on* E .

*Suppose*

*i)* $\{a_0\}^c$ *is commutative , semi-simple and symmetric with respect to* $^\sim$

*ii)* $PSp(C_{f(0)}) \cap \mathbb{N} = \emptyset$

*Then we have (for* $\sigma_0$ *real) :*

*a)* $\mathscr{A}$ *is commutative, semi-simple and symmetric.*

*b)* $\Gamma_0 : \mathscr{A} \to \{a_0\}^c$ *is an injective *-homomorphism.*

*c)* $q_0$ *is unitary relative to* $^\sim$ *with* $Sp(q_0) \subset S^1$ *and there are selfadjoint* u,v $\in \mathscr{A}$ *with :*

$q_0 = u + iv$ *and* $\Gamma_0(u) = \cos(2\pi a_0)$ , $\Gamma_0(v) = \sin(2\pi a_0)$ , $u^2 + v^2 = \mathbf{e}$ .

Let us conclude this section with a final

**Remark 5.12 :**

Let $\mathfrak{V}$ be any closed subalgebra of $\mathscr{B}$ containing $\mathfrak{F}$ and $\{\mathbf{e}\}$. Then all coefficients of the Taylor series of f at 0 belong to $\mathfrak{V}$ ; if Rea $\sigma_0 < -2\pi$ the method of Cauchy (Lemma 3.1) resp. Meyer-Hamburger in constructing $v_0$ shows, that $\exp(2\pi i a_0)$ , $q_0$ und $q_0^{-1}$ all belong to $\mathfrak{V}$ . Moreover $\mathfrak{W}, \mathfrak{W}^{-1} \subset \mathfrak{V}$ , $\{g_0(\zeta) : \zeta \in E\} \subset \mathfrak{V}^*_0$.

## 6. Factors of Automorphy with Logarithms

Proposition 5.3 would provide a positive answer to (P1), if the question whether an element x of $\mathscr{B}$ belongs to $\exp(\mathscr{B})$ only was dependent on the spectrum $Sp(x)$ . But such a characterization is not possible in general:
Consider in $\mathscr{X} := C([0,1])$ the subalgebra $\mathfrak{T}$ of functions h with $h(0) = h(1)$. $\mathfrak{T}$ is clearly a full subalgebra of $\mathscr{X}$.Then for $\tau \in C([0,1])$, $\tau(t) = t$ , $t \in [0,1]$ , we have $\gamma := \exp(2\pi i\tau) \in \mathfrak{T}$ and $Sp_{\mathscr{X}}(\gamma) = Sp_{\mathfrak{T}}(\gamma) = S^1$ , but $\gamma \in \exp(\mathscr{X})$ and $\gamma \notin \exp(\mathfrak{T})$ .

**Remark:** By the way, this example elucidates the fact, that in a commutative, semi-simple symmetric unital Banach algebra $\mathscr{X}$ (not necessarily a B*-algebra!) we have $\mathscr{X}^*$ connected, if the Gelfand space $X(\mathscr{X})$ is metrizable and contractible.

Let $\mathfrak{P} := \{ q \in \mathscr{B} : \exp(2\pi iq) = \mathbf{e} \}$ be the set of logarithms of unity in $\mathscr{B}$, i.e. $\mathfrak{P}$ is the set of finite integer linear combinations of pairwise orthogonal idempotents (cf. section 3). It is elementary functional calculus to show, that if $0 \in R_0(q_0)$, we have $q_0 \in \exp(\mathscr{B})$.
If $q_0 \in \exp(\mathscr{B})$ let $x \in \mathscr{B}$ so, that $q_0 = \exp(2\pi ix)$; then obviously $x \in \{x\}^c \subset \{q_0\}^c$ .
In our situation we get slightly more :

**Proposition 6.1:**
*a) If* $0 \in R_0(q_0)$ *, then* $q_0 \in \exp(\mathscr{A})$.
*b) If* $0 \in R_0(\exp(2\pi ia_0))$ *, then* $q_0 \in \exp(\mathscr{A})$.
*c) The following statements are equivalent :*
*i) There is a* $y_0 \in \mathscr{A}$ *with :*
$$\Gamma_0(y_0) - a_0 \in \mathfrak{P} \cap \{a_0\}^c .$$
*ii)* $q_0 \in \exp(\mathscr{A})$.

**Proof:**
a) Let $x \in \mathscr{B}$ with $q_0 = \exp(2\pi ix)$ be defined via functional calculus, that is $x = 1/(2\pi i) \int_C \ln(\lambda) (\lambda - q_0)^{-1} d\lambda$ , where integration takes place over a suitable contour surrounding $Sp(q_0)$ once in the positive sense and with a suitable single-valued branch of $\ln(\lambda)$ inside the contour.
Obviously $x \in \{q_0\}^c$. For $\sigma \in H$:

$$p^*g_0(\sigma) = v_0(\sigma)\ q_0\ v_0(\sigma)^{-1} = v_0(\sigma)\ \exp(2\pi ix)\ v_0(\sigma)^{-1} = \exp(2\pi i v_0(\sigma)\ x v_0(\sigma)^{-1}) =$$
$$= \exp(2\pi i p^*h_x(\sigma)),$$

thus $g_0 = \exp(2\pi ih_x)$ (interpreted pointwise or uniformly on a compact subset of E containing 0 ).

Since $Sp(g_0(\zeta)) = Sp(q_0)$ for $|\zeta| > 0$, we get :

$2\pi i h_x(\zeta) = \int_C \ln(\lambda)\,(\lambda - g_0(\zeta))^{-1}\,d\lambda$.

If we restrict $\zeta$ to a compact neighborhood of 0 , it is easily seen that $h_x(\zeta)$ remains bounded as $\zeta$ approaches 0 . Thus $x \in \mathscr{A}$.

b) With a and Proposition 5.3 .

c) ii$\Rightarrow$ i: Let $y_0 \in \mathscr{A}$ with $q_0 = \exp(2\pi i y_0)$.

Then $\exp(2\pi i a_0) = \Gamma_0(q_0) = \exp(2\pi i\,\Gamma_0(y_0))$ ; since $\Gamma_0(y_0)$ and $a_0$ commute, we get

$\Gamma_0(y_0) - a_0 \in \mathfrak{P} \cap \{a_0\}^c$ .

i$\Rightarrow$ ii:

Let $y_0 \in \mathscr{A}$ with $\Gamma_0(y_0) - a_0 \in \mathfrak{P}$. Then:

$\Gamma_0(\exp(2\pi i y_0)) = \exp(2\pi i a_0) = \Gamma_0(q_0)$ ,

$\exp(2\pi i y_0) - q_0 =: n$,

with $n \in \mathscr{A} \subset \{q_0\}^c$ ( 5.1 e and g ).

Let $\mathscr{C}$ be a maximal commutative subalgebra of $\mathscr{A}$ containing $y_0$ , $q_0$ and $\mathbf{e}$ , then $n \in \mathscr{C}$ and $\exp(2\pi i y_0) \in (\mathscr{C}^*)_0$ .

With n we have $n' := q_0^{-1} n$ nilpotent and from

$$\mathbf{e} + n' = q_0^{-1} \exp(2\pi i y_0)$$

and

$$\mathbf{e} + n' \in (\mathscr{C}^*)_0 \quad (\text{ since } Sp(\mathbf{e} + n') = \{1\})$$

we get

$$q_0^{-1} \in (\mathscr{C}^*)_0 \quad \text{, i.e.}$$

$$q_0 \in (\mathscr{C}^*)_0 \quad , \quad q_0 \in \exp(\mathscr{A}). \qquad \text{q.e.d}$$

The existence of a logarithm of $q_0$ has - as already indicated in section 2 - clear consequences for the structure of solutions of (*):

**Proposition 6.2:**

*A) The following statements are equivalent :*

*i) There is a* $y_0 \in \mathscr{B}$ *with* $q_0 = \exp(2\pi i y_0)$.

*ii) There are a* $y_0 \in \mathscr{B}$ *and a function* $m \in \mathscr{M}^{\mathscr{B}^*}(E)$ *with* $v_0(\sigma) = p^* m(\sigma) \exp(y_0(\sigma - \sigma_0))$ *on* H.

*B) If* $\{a_0\}^c$ *contains no non-trivial idempotents, then we have equivalence between :*

*iii) There is* $y_0 \in \mathscr{A}$ *with* $q_0 = \exp(2\pi i y_0)$.

*iv) There are a* $y_0 \in \mathscr{A}$ *and* $m \in \mathscr{O}^{\mathscr{B}^*}(E)$ *with* $v_0(\sigma) = p^* m(\sigma) \exp(y_0(\sigma - \sigma_0))$ *on* H.

*C) If* $q_0 \in \exp(\mathscr{A})$ *then there are* $r \in \mathfrak{P} \cap \{a_0\}^c$ , $y \in \mathscr{B}$ *and* $h \in \mathscr{O}^{\mathscr{B}^*}(E)$ *with* $h(0) = \mathbf{e}$ *such that* $v(\sigma) = p^* h(\sigma) \exp(-r\,\sigma) \exp(y\,\sigma)$ , $\sigma \in H$, *is a fundamental solution of (*).*

*D) If there is* $y_0 \in \mathscr{B}$ *with* $q_0 = \exp(2\pi i y_0)$ *and* $\mathscr{A} = \{y_0\}^c$ *, then there are a* $z \in \mathscr{B}^*_0$ *and finitely many pairwise orthogonal idempotents* $p_i \in \{a_0\}^c$ *with* $\sum p_i = \mathbf{e}$ *such that :*

$$\Gamma_0 = \sum {}^{L}p_i\, {}^{R}p_i\, \alpha_z \mid_{\mathscr{A}} .$$

**Remark:** *The condition of C) is always satisfied in an algebra $\mathscr{B}$ where each element has its spectrum totally disconnected. This completes the proof of Theorem 1.3 .*
*For the matrix case 6.2 C is essentially the classical structure theorem for fundamental solutions of (*) (cf. [Ga], Vol.2, pp.148-164).*

**Proof:**

A) a) ii$\Rightarrow$ i trivial.

b) i$\Rightarrow$ ii: Obviously there is a m $\in \mathscr{O}^{\mathscr{B}*}(E^*)$ with $v_0(\sigma) = p^*m(\sigma) \exp(y_0(\sigma-\sigma_0))$ on H;
but m satisfies on E* the differential equation:
$\zeta\, m'(\zeta) = f(\zeta)\, m(\zeta) - m(\zeta)\, y_0$.
With Lemma 3.4 we see, that m has at most a pole in 0 and we can argue analogously for $m^{-1}$.

B) c) iv$\Rightarrow$ iii trivial again.

d) iii$\Rightarrow$ iv: As above we have $m \in \mathscr{O}^{\mathscr{B}*}(E^*)$ with $v_0(\sigma) = p^*m(\sigma) \exp(y_0(\sigma-\sigma_0))$ on H; m and $m^{-1}$ have at most poles in 0 . But using Proposition 6.1c we can assume, that $y_0$ is chosen such that $\Gamma_0(y_0) = a_0$ .

Consider the differential equation for m:
$\zeta\, m'(\zeta) = f(\zeta)\, m(\zeta) - m(\zeta)\, y_0$,
and let $h := \Phi^{-1}(y_0) \in \mathscr{O}^{\mathscr{B}}(E)$, $h(0) = a_0$ . Then $h(\zeta) = m(\zeta)\, y_0 m(\zeta)^{-1}$.

Thus:
$\zeta\, m'(\zeta)\, m(\zeta)^{-1} = f(\zeta) - h(\zeta)$ ,
$\zeta\, m'(\zeta) = (f(\zeta) - h(\zeta))\, m(\zeta)$ ,
$\zeta\, (m^{-1})'(\zeta) = -m^{-1}(\zeta)\, (f(\zeta) - h(\zeta))$ .
But according to the choice of $y_0$ the function $1/\zeta\, (f(\zeta) - h(\zeta))$ h is holomorphic in E, and Lemma 3.1 tells us, that m and $m^{-1}$ are both holomorphic in E.

C)
Take a $y_0 \in \mathscr{A}$ with $r := \Gamma_0(y_0) - a_0 \in \mathfrak{P} \cap \{a_0\}^c$ (Proposition 6.1) and write
$v_0(\sigma) = p^*m(\sigma) \exp(y_0(\sigma-\sigma_0))$, $\sigma \in H$, $m \in \mathscr{O}^{\mathscr{B}*}(E^*) \cap \mathscr{M}^{\mathscr{B}*}(E)$ .
With $k := h_{y_0} \in \mathscr{O}^{\mathscr{B}}(E)$ we see that m is a normalized fundamental solution of
(+) $\zeta m'(\zeta) = (f(\zeta) - k(\zeta))\, m(\zeta)$ ,
which additionally is single-valued in E.
The leading coefficient ( "$a_0$") of (+) is -r .

Lemma 3.7 ii shows that (+) satisfies Hille's condition in order to get a Frobenius solution of (+) ( [Hi], Theorem 6.6.1, p. 234).

But since m is single-valued, Proposition 2.2 shows us, that a Frobenius solution of (+) is necessarily of type 1.
Thus we get a $h \in \mathcal{O}^{\mathcal{B}*}(E)$ with $h(0) = \mathbf{e}$ and $z \in \mathcal{B}^*_0$ with
$p^*m(\sigma) = p^*h(\sigma) \exp(- r\ \sigma)\ z$ ; set $y := z\, y_0\, z^{-1}$ .

D) We will continue the arguments of C).
If we write (according to Hille's result on logarithms of the unit , [BD], p. 42):
$r = n_1 p_1 + .... + n_N p_N$ with $n_i \in \mathbb{Z}$ and the $p_i$ pairwise orthogonal idempotents from $\{a_0\}^c$ which sum up to $\mathbf{e}$ , we may suppose (by adding an integer multiple of $\mathbf{e}$ to $y_0$) , that:

i) the $n_i$ are all distinct, i.e. $n_i < n_k$ if $i < k$

ii) $n_N < 0$ .

Since $\exp(- r\ \sigma)$ is single valued in E, the order of m in 0 is exactly $- n_1 > 0$ with leading coefficient $p_1 z$ .

So for $x \in \mathcal{A} = \{y_0\}^c$ we get

$h_x(\zeta) = h(\zeta)\ [\ \sum \zeta^{n_j - n_i} p_i\, \alpha_z(x)\, p_j\ ]\ h(\zeta)^{-1}$.

The expression $\sum \zeta^{n_j - n_i} p_i\, \alpha_z(x)\, p_j$ is holomorphic in 0,

hence
$0 = p_i\, \alpha_z(x)\, p_j$ for $i > j$ and ( recall $h(0) = \mathbf{e}$ )

$\Gamma_0(x) = \sum p_i\, \alpha_z(x)\, p_i$ .

q.e.d.

For a finite-dimensional $\mathcal{B}$ the additional assumption of 6.2 B , that $\mathfrak{P} \cap \{a_0\}^c = \mathbb{Z}\, \mathbf{e}$ (cf. Lemma 3.7) implies, that $a_0$ is of the form $a_0 = \lambda \mathbf{e} + n$ with nilpotent n . Thus the commutator corresponding to $a_0$ is itself nilpotent. On the other hand: the Jordan blocks of $a_0$ are exactly of this form.

Since there are non-commutative unital Banach algebras without divisors of zero, let us note more generally:

**Proposition 6.3:** *If* $\mathcal{B}$ *contains no nilpotent elements then* $\mathcal{A} = \{q_0\}^c$ *and the following statements are equivalent :*

*i) There is a* $y_0 \in \mathcal{B}$ *with* $q_0 = \exp(2\pi i y_0)$.

*ii) There are* $y_0 \in \mathcal{B}$ *and* $m \in \mathcal{O}^{\mathcal{B}*}(E)$ *with* $v_0(\sigma) = p^*m(\sigma) \exp(y_0(\sigma - \sigma_0))$ *on* H.

*iii) There is a* $z \in \mathcal{B}^*$ *with* $q_0 = \alpha_z(\exp(2\pi i a_0))$.

*iv)* $a_0 \in \Gamma_0(\mathcal{A})$ .

*v)* $\Gamma_0 : \mathcal{A} \to \{a_0\}^c$ *is bijective.*

*vii) (*) has a Frobenius solution.*

**Proof:**

Eigenvectors of a commutator are nilpotent, 5.1 yields $\mathcal{A} = \{q_0\}^c$.

a) iii$\Rightarrow$ i trivial.

b) i$\Rightarrow$ ii : We have $q_0 \in \exp(\mathcal{A})$. But the idempotents provided by 6.1 are central (cf. 4.2), thus contained in the image of $\Gamma_0$.

Argue as in 6.2 B d .

c) ii$\Rightarrow$ iii: Set z := m(0).

d) The equivalence of ii with iv and v follows from Proposition 6.10. (see below).

e) ii$\Rightarrow$ vii: Proposition 6.4 (see below).

f) vii$\Rightarrow$ ii: Proposition 2.2.

q.e.d.

Back to a general $\mathcal{B}$ we get:

**Proposition 6.4:** *The following statements are equivalent :*

*i) There are a* $y_0 \in \mathcal{B}$ *and a* $m \in \mathcal{O}^{\mathcal{B}*}(E)$ *with* $v_0(\sigma) = p^*m(\sigma) \exp(y_0(\sigma-\sigma_0))$ *on* H.

*ii) There is a fundamental solution* v *of (*) of the form :*

$$v(\sigma) = p^*h(\sigma) \exp(a_0 \sigma) \text{ on } H \text{ with } h \in \mathcal{O}^{\mathcal{B}*}(E) \text{ and } h(0) = \mathbf{e} .$$

*iii) Any fundamental solution of (*) is of the form :*

$$w(\sigma) = p^*g(\sigma) \exp(z(\sigma-\sigma_0)) \text{ on } H \text{ with } z \in \mathcal{B} \text{ and } g \in \mathcal{O}^{\mathcal{B}*}(E) .$$

*If any of i-iii holds, then :*

*iv)* $\Gamma_0 : \mathcal{A} \to \{a_0\}^c$ *is surjective.*

**Proof:**

a) ii$\Rightarrow$ i : There is a $z \in \mathcal{B}^*$ with $v = v_0 z$, thus $z = h(\zeta_0)\exp(a_0 \sigma_0)$ and:

$$\begin{aligned} v_0(\sigma) &= p^*h(\sigma) \exp(a_0 \sigma) z^{-1} \\ &= p^*h(\sigma)\exp(a_0 \sigma_0)z^{-1}z \exp(a_0 (\sigma-\sigma_0)) z^{-1} \\ &= p^*h(\sigma)\exp(a_0 \sigma_0)z^{-1} \exp(a_z(a_0)(\sigma-\sigma_0)) \\ &= p^*h(\sigma)\, h(\zeta_0)^{-1} \exp(a_z(a_0)(\sigma-\sigma_0)) . \end{aligned}$$

Let $m := hh(\zeta_0)^{-1}$ and $y_0 := a_z(a_0)$.

b) i$\Rightarrow$ ii : Let $z := m(0) \in \mathcal{B}^*$ .

Then $\Gamma_0(q_0) = \alpha_z(q_0) = \exp(2\pi i a_0) = \alpha_z(\exp(2\pi i y_0))$,

$\alpha_z : \{q_0\}^c \to \{\exp(2\pi i a_0)\}^c$ is an isomorphism.

From the differential equation of m: $\zeta m'(\zeta) = f(\zeta)m(\zeta) - m(\zeta)y_0$ we get for $\zeta \to 0$ :

$$a_0 = \alpha_z(y_0) ,$$

i.e. equally

$$\alpha_z : \{y_0\}^c \to \{a_0\}^c$$

is an isomorphism.

Let $u := v_0 z^{-1}$; then u is fundamental solution of (*) with $\exp(2\pi i a_0)$ as factor of automorphy.

Let g be defined by $u(\sigma) = p^*g(\sigma)\exp(a_0\,\sigma)$, $g \in \mathcal{O}^{\mathcal{B}^*}(E^*)$ .

Then:

$u(\sigma) = p^*g(\sigma)\exp(a_0\,\sigma) = p^*g(\sigma)\, z\exp(y_0\,\sigma_0)\exp(y_0\,(\sigma-\sigma_0))\, z^{-1}$ , i.e.:

$m = g\, z\exp(y_0\,\sigma_0)$, in particular $g \in \mathcal{O}^{\mathcal{B}^*}(E)$.

Hence $g(0) = \exp(-a_0\,\sigma_0)$; with $h := g\exp(a_0\,\sigma_0)$, $v := u\exp(a_0\,\sigma_0)$ we get :

$v = u\exp(a_0\,\sigma_0) = p^*g(\sigma)\exp(a_0\,\sigma)\exp(a_0\,\sigma_0) = p^*h(\sigma)\exp(a_0\,\sigma)$ .

c) iii $\Rightarrow$ i trivial.

ii $\Rightarrow$ iii: Let w be an arbitrary fundamental solution of (*). There is a $x \in \mathcal{B}^*$ with:

$w(\sigma) = v_0(\sigma)\,x = p^*m(\sigma)\exp(y_0(\sigma-\sigma_0))\,x = p^*m(\sigma)\,x\exp(x^{-1}y_0\,x(\sigma-\sigma_0))$ .

Let $z := x^{-1}y_0\,x$ and $n := mx \in \mathcal{O}^{\mathcal{B}^*}(E)$ .

d) ii $\Rightarrow$ iv :

We have a fundamental solution $v_1$ of (*) of the form:

$v_1(\sigma) = p^*h(\sigma)\exp(a_0(\sigma-\sigma_0))$ , $\sigma \in H$, with $h \in \mathcal{O}^{\mathcal{B}^*}(E)$ .

Set $x := v_1(\sigma_0) \in \mathcal{B}^*$ , i.e. $v_1 = v_0\,x$ and therefore $\exp(2\pi i a_0) = \alpha_x^{-1}(q_0)$.

$\alpha_x : \{\exp(2\pi i a_0)\}^c \to \{q_0\}^c$ is an isomorphism,

$\{a_0\}^c \subset \{\exp(2\pi i a_0)\}^c$ .

For $y \in \{q_0\}^c$ and $\sigma \in H$ we get with $g := p^*h$

(+) $p^*h_y(\sigma) = v_0(\sigma)\,y\,v_0(\sigma)^{-1} = v_1(\sigma)x^{-1}yx\,v_1(\sigma)^{-1} = \alpha_{g(\sigma)}\exp(C_{f(0)}(\sigma-\sigma_0))\alpha_x^{-1}(y)$.

$\alpha_{g(\sigma)}$ is the lifting of a function holomorphic in 0, for $y \in \{q_0\}^c$ with $\alpha_x^{-1}(y) \in \{a_0\}^c$ we see thus that $h_y$ is holomorphic in 0 too, hence

$\alpha_x(\{a_0\}^c) \subset \mathcal{A}$.

With $z := h(0) \in \mathcal{B}^*$ we get from $\zeta h'(\zeta) = f(\zeta)h(\zeta) - h(\zeta)a_0$ , $f(0) = a_0$, $\zeta \in E$ :

$a_0 z = z a_0$, hence:

$\alpha_z : \{a_0\}^c \to \{a_0\}^c$ is an isomorphism.

(+) shows us, that $\alpha_z$ equals the restriction of $\Gamma_0 \circ \alpha_x$ to $\{a_0\}^c$ .

If now $a \in \{a_0\}^c$ , we get a $b \in \{a_0\}^c$ with $a = \alpha_z(b)$. Let $y := \alpha_x(b)$, i.e. $y \in \mathcal{A}$ and :

$\Gamma_0(y) = \Gamma_0(\alpha_x(b)) = \alpha_z(b) = a$,

$\Gamma_0 : \mathcal{A} \to \{a_0\}^c$ is onto . q.e.d.

While for $q_0 \in \exp(\mathcal{A})$ a weak form of surjectivity of $\Gamma_0$ is sufficient (6.1) , 6.4 shows that a positive answer to P2 already implies $\Gamma_0$ to be surjective. From 6.2 and 6.4 we see that a suitable condition - however not so interesting in the matrix case - on $\{a_0\}^c$ forces these two forms of surjectivity to be equivalent.

Propositions 4.1 and 5.2 show that $q_0 \in \exp(\mathcal{B})$, if $\mathrm{Sp}(C_{f(0)}) \cap \mathbb{Z} = \{0\}$ .

However, this condition is far from being necessary ( take $\mathscr{B} = M_n(\mathbb{C})$ ), on the other hand it is so strong, that it implies much more.

A detailed look at the proof of Theorem 6.5.3 in [Hi] shows, that Hille proves essentially assertion ii of the following Proposition, which can be regarded as an abstract analogue of the classical lemma of H. Kneser.

**Proposition 6.5:** *Let* $f : E \to \mathscr{B}$ *be holomorphic with* $f(\zeta) = \sum_{k=0}^{\infty} a_k \zeta^k$, $a_k \in \mathscr{B}$.

*We consider the differential equation :*

(+) $\zeta h'(\zeta) = f(\zeta)\, h(\zeta) - h(\zeta)\, a_0$, $\zeta \in E$.

*i) If* $PSp(C_{f(0)})$ *contains no natural numbers, then a single-valued solution of (+) in* E *with* $h(0) = \mathbf{e}$ *is unique. Under this assumption k-invariance of* f *implies k-invariance of* h.
*ii)*

*If* $h(\zeta) = \sum_{k=0}^{\infty} c_k \zeta^k$, $c_k \in \mathscr{B}$, $c_0 = \mathbf{e}$, *is a formal solution of (+),*

*then the series representing* h *converges in* E *and provides an actual solution of (+).*

**Proof:**
a) Let $PSp(C_{f(0)}) \cap \mathbb{N} = \varnothing$ h a solution of (+) with $h(0) = 0$, then there are $m \in \mathbb{N}$ and $g : E \to \mathscr{B}$ holomorphic with $g(0) \neq 0$, such that $h(\zeta) = \zeta^m g(\zeta)$ for $\zeta \in E$.
Substituting this in (+) and letting $\zeta$ tend to 0, we see that $g(0)$ is an eigenvector of $C_{f(0)}$ corresponding to the eigenvalue m.
Thus solutions of (+), holomorphic in the whole of E, are uniquely determined by their value in 0.

b) If $\eta$ is a m^th root of unity and h a single-valued solution of (+) holomorphic in E, so is $g(\zeta) := h(\eta\zeta)$ with $g(0) = h(0)$, then by uniqueness $h = g$, i.e. h is m-invariant.
c) We can adopt literally the argumentation of Hille.
h is a formal solution of (+) if and only if we have for the coefficients $c_k$ :

$$(k\,\mathbf{id} - C_{f(0)})(c_k) = \sum_{l=0}^{k-1} a_{k-l}\, c_l \quad \text{for } k > 0 .$$

For $k > 2\,\|a_0\| \geq \|C_{f(0)}\|$ it follows with $\alpha_k := \|a_k\|$ :

$$(k - 2\alpha_0)\ \|c_k\| \ \leq \sum_{l=0}^{k-1} \alpha_{k-l}\ \|c_l\|\ .$$

Let $0 < r < 1$ and

$$F(r) := \sum_{k=0}^{\infty} \alpha_k\ r^k\ ,$$

then F is continuous in r.
If we fix r, it will be sufficient to show that the set $\{\ \|c_k\|\ r^k\}$ is bounded, because then the series for h converges for $|\zeta| < r$ , but $r < 1$ is arbitrary.
With $M_k := \max\{\ \|c_l\|\ r^l : l = 0, ..., k\}$ we get
$\|c_k\|\ r^k \leq M_{k-1}\ (\ F(r) - \alpha_0\ )\ /\ (\ k - 2\alpha_0\ )$ , thus for sufficiently large k the sequence $(M_k)$ becomes stationary, hence is bounded. q.e.d.

Assertion i of the following Proposition is Theorem 6.5.3 in [Hi]; assertion ii is proved alternatively - in an equivalent (by 5.1) formulation for solutions of (**) - by Schäfke and Schmidt [Sch].

**Proposition 6.6:** *If* $\mathrm{Sp}(C_{f(0)}) \cap \mathbb{N} = \emptyset$ *,then the following holds :*
*i) There is a fundamental solution* v *of (*) of the form :*
$v(\sigma) = p^*h(\sigma)\exp(a_0\,\sigma)$ *on* H *with* $h \in \mathscr{O}^{\mathscr{B}^*}(E)$ *and* $h(0) = \mathbf{e}$ .
*ii)* $\Gamma_0 : \mathscr{A} \to \{a_0\}^c$ *is bijective.*

**Proof:**
a) i: The proof of 6.5 shows that the assumption $\mathrm{Sp}(C_{f(0)}) \cap \mathbb{N} = \emptyset$ immediately provides a formal solution h of (+) with $h(0) = \mathbf{e}$ , which by 6.5 is an actual solution .
One verifies, that v defined by i is a solution of (*) , which is fundamental by Lemma 3.1.

b) ii: From i and Proposition 6.4 we get: $\Gamma_0 : \mathscr{A} \to \{a_0\}^c$ is surjective. Injectivity follows from 5.1. q.e.d.

We will see below, that the statements i,ii of 6.6 are even equivalent.

**Corollary 6.7:** *If* $\mathrm{Sp}(C_{f(0)})$ *contains no integers different from* 0, *then* $\mathscr{A} = \{q_0\}^c$ *and* $\Gamma_0 : \mathscr{A} = \{q_0\}^c \to \{a_0\}^c$ *is an isomorphism.*

The assumptions of 6.6 or 6.7 are especially satified, if $a_0$ is sufficiently close to the center of $\mathscr{B}$, e.g. if $d(a_0, \mathscr{B}^c) < ½$. But again, spectral properties of $a_0$ are not exclusively decisive for the existence of fundamental solutions of the form as in 6.6 i :

**Proposition 6.8:** *If* f *is k-invariant with* $k >$ diam (Rea(Sp($a_0$))) *, then (*) has a fundamental solution of the form* $v(\sigma) = p^*h(\sigma)\exp(a_0\sigma)$ *with* $h \in \mathscr{O}^{\mathscr{B}*}(E)$ *k-invariant and* $h(0) = \mathbf{e}$ .

**Proof:** We set for $\sigma \in H$ : $v_k(\sigma) := v_0(\sigma/k)$ , i.e. $v_k(k\sigma_0) = \mathbf{e}$ and see

(+) $v_k'(\sigma) = 1/k\, f(e^{\sigma/k})\ v_k(\sigma)$ .

The formula of Cauchy-Hadamard for the radius of convergence of a power series tells us, that $f_k(\zeta) := 1/kf(\zeta^{1/k})$ is holomorphic in E. With $a' := f_k(0) = a_0/k$ we get by assumption: diam(Rea(Sp($C_{a'}$)) $< 1$ , that is. Sp($C_{a'}$) contains no integers different from 0.

From 6.6 we get a fundamental solution u of (+) of the form $u(\sigma) = p^*h(\sigma)\exp(a_0/k\ \sigma)$ with $h \in \mathscr{O}^{\mathscr{B}*}(E)$ and $h(0) = \mathbf{e}$ .

For $v(\sigma) := u(k\sigma)$ it follows $v(\sigma) = p^*h(k\sigma)\exp(a_0\sigma)$

and $v'(\sigma) = ku'(k\sigma) = f(e^{\sigma})\ v(\sigma)$ . q.e.d.

**Remark:** Proposition 6.5 provides an alternative, more direct proof of 6.8.

Proposition 6.8 tells us in particular, that for an arbitrary $a_0 \in \mathscr{B}$ there is a differential equation (*) with a non-constant function f ( set $f(\zeta) = a_0 + b\zeta^k$ with b not commuting with $a_0$ and k sufficiently large ) , which admits of a fundamental solution v of the form $v(\sigma) = p^*h(\sigma)\exp(a_0\sigma)$ with $h \in \mathscr{O}^{\mathscr{B}*}(E)$ and $h(0) = \mathbf{e}$ .

Thus for $q_0 \in \exp(\mathscr{B})$ it is definitely not necessary that $\exp(2\pi i a_0)$ be an interior point of $\exp(\mathscr{B})$ .

The following Proposition summarizes some conclusions, which are idependent of 4.1 and 6.6:

**Proposition 6.9:** *If* $q_0 \in \exp(\mathscr{B})$ *and* PSp( $C_{f(0)}$ ) $\cap\ \mathbb{Z} \setminus \{0\} = \varnothing$ *, then :*

*i)* $\mathscr{A} = \{q_0\}^c$ , $q_0 \in \exp(\mathscr{A})$ .

*ii) If* $y_0 \in \mathscr{A}$ *with* $q_0 = \exp(2\pi i y_0)$ *, then* $\{y_0\}^c \subset \mathscr{A} = \{q_0\}^c$.

*iii)* $\Gamma_0 : \mathscr{A} = \{q_0\}^c \to \{a_0\}^c$ *is injective,* $\mathrm{Sp}(x) = \mathrm{Sp}(\Gamma_0(x))\ \forall\ x \in \{q_0\}^c$ ,

*especially* $\mathrm{Sp}(q_0) = \mathrm{Sp}(\exp(2\pi i a_0))$ .

*iv)* $r := \Gamma_0(y_0) - a_0 \in \mathfrak{P} \cap \{a_0\}^c$ , $\Gamma_0\ C_{y0} = C_r\ \Gamma_0$ *on* $\{q_0\}^c$.

*v) There is a* $m \in \mathscr{M}^{\mathscr{B}*}(E)$ *with* $v_0(\sigma) = p^*m(\sigma)\exp(y_0(\sigma-\sigma_0))$ *on* H.

*vi) For* $z \in \{y_0\}^c$ *we have* $h_z(\zeta) = \alpha_{m(\zeta)}(z)$ *holomorphic in 0.*

*vii) If* $r \in \{a_0\}^{cc}$ *, then* $\{y_0\}^c = \mathscr{A} = \{q_0\}^c$.

*viii)* $\Gamma_0(\{q_0\}^{cc}) = \Gamma_0(\mathscr{A}) \cap \Gamma_0(\mathscr{A})^c$.

*ix) If* $\{a_0\}^c$ *is commutative, then* $\{y_0\}^c = \mathcal{A} = \{q_0\}^c$ *is commutative.*

*x) If* $\mathfrak{P} \cap \{a_0\}^c = \mathbb{Z}\mathbf{e}$ *, then* $a_0 \in \Gamma_0(\mathcal{A})$ .

*xi)* $a_0 \in \Gamma_0(\mathcal{A}) \Leftrightarrow m \in \mathcal{O}^{\mathcal{B}^*}(E) \Leftrightarrow \Gamma_0 : \mathcal{A} = \{q_0\}^c \to \{a_0\}^c$ *is bijective.*

The last assertion is true under much weaker conditions; this implies that the statements i, ii of Proposition 6.6 are indeed equivalent:

**Proposition 6.10:** *If* $\mathrm{PSp}( C_{f(0)} ) \cap \mathbb{N} = \varnothing$ *, then the followings statements are equivalent :*

*i)* $a_0 \in \Gamma_0(\mathcal{A})$ .

*ii) There are* $y_0 \in \mathcal{A}$ *with* $q_0 = \exp(2\pi i y_0)$.

*and* $m \in \mathcal{O}^{\mathcal{B}^*}(E)$ *with* $v_0(\sigma) = p^*m(\sigma) \exp(y_0( \sigma-\sigma_0))$ *on* H.

*iii)* $\Gamma_0 : \mathcal{A} \to \{a_0\}^c$ *is bijective.*

*If additionally* $\mathrm{PSp}( C_{f(0)} ) \cap - \mathbb{N} = \varnothing$ *, then these statements are equivalent to :*

*iv) There is a* $x \in \mathcal{B}^*$ *with* $q_0 = x \exp(2\pi i a_0) x^{-1}$ .

**Proof:**

The assumption says essentially that $\Gamma_0 : \mathcal{A} \to \{a_0\}^c$ is injective.

a) iii ⇒ i: trivial.

b) i ⇒ ii: There is a $y \in \mathcal{A}$ with $a_0 = \Gamma_0 (y)$ , i.e.

$$\exp(2\pi i a_0) = \exp(2\pi i\, \Gamma_0 (y) ) = \Gamma_0 ( \exp(2\pi i y) ) = \Gamma_0 ( q_0 ).$$

As $\Gamma_0$ is injective: $q_0 = \exp(2\pi i y)$.

The statement concerning m follows as in the proof of Proposition 6.2.

c) ii ⇒ iii: By assumption and Proposition 6.4.

d) ii ⇒ iv: Set $x := m(0)$ .

e) iv ⇒ iii: The fundamental solution $v := v_0 x$ has as factor of automorphy: $\exp(2\pi i a_0)$; hence there is a $m \in \mathcal{O}^{\mathcal{B}^*}(E^*) \cap \mathcal{M}^{\mathcal{B}^*}(E)$ with $v := p^*m \exp(a_0\sigma )$. As $C_{f(0)}$ has no integer eigenvalues different from 0, $m$, $m^{-1}$ are both holomorphic in 0. From Proposition 6.4 follows:

$\Gamma_0 : \mathcal{A} \to \{a_0\}^c$ is surjective. q.e.d.

**Corollary 6.11:** *Suppose* $\mathrm{PSp}( C_{f(0)} ) \cap \mathbb{N} = \varnothing$ *and* $a_0 \in \Gamma_0(\mathcal{A})$ .

*If* $\|\exp(\pm 2\pi i a_0) \| = 1$, *then the group* $\{q_0^k : k \in \mathbb{Z} \}$ *is bounded and* $\mathrm{Sp}(q_0) \subset S^1$ .

**Proof:** By assumption, we are in the circumstances of 6.10, where assertion iv holds.
For $k \in \mathbb{Z}$ we have $q_0^k = \alpha_x^{-1}(\exp(2\pi i k a_0))$ , i.e.
$\|q_0^k \| \leq \| \alpha_x^{-1} \|$ . q.e.d.

As summary of propositions 5.6, 6.2 and 6.9 let us formulate:

**Proposition 6.12:** *If* $q_0 \in \exp(\mathscr{B})$ *and* $\mathrm{PSp}(C_{f(0)}) \cap \mathbb{Z} = \{0\}$ , *and if* $\{a_0\}^c$ *contains no non-trivial idempotents, then* :

*i) For* $y_0 \in \mathscr{B}$ *with* $q_0 = \exp(2\pi i y_0)$ *there is a* $m \in \mathscr{O}^{\mathscr{B}*}(E)$ *with* $v_0(\sigma) = p^*m(\sigma)\exp(y_0(\sigma-\sigma_0))$ *on* H.

*ii)* $\Gamma_0 : \mathscr{A} \to \{a_0\}^c$ *ist bijective.*

*iii) If additionally* $\{a_0\}^c$ *is commutative, then* $\mathscr{A} = \{q_0\}^c = \{y_0\}^c$ *and with* $x := m(0)$ *we have*

$$\Gamma_0 = {}^{\alpha}x \mid \{q_0\}^c , \ x \in \mathfrak{F}^{cc}.$$

The proof of Proposition 6.4 iv turned out to be quite cumbersome; we shall now make full use of the findings of section 4, to get some more insight in the structrure of $\mathscr{A}$ and to streamline some arguments.

First let us prove a complement to Proposition 6.10:

**Proposition 6.13 :** *Suppose* $\Gamma_0 : \mathscr{A} \to \{a_0\}^c$ *bijective. Then :*

*i) There are* $y_0 \in \mathscr{A}$ *with* $q_0 = \exp(2\pi i y_0)$
*and* $m \in \mathscr{O}^{\mathscr{B}*}(E)$ *with* $v_0(\sigma) = p^*m(\sigma)\exp(y_0(\sigma-\sigma_0))$ *on* H.

*ii)* $\mathrm{PSp}(C_{f(0)}) \cap \mathbb{N} = \varnothing$ .

**Proof:**

i) There is a $y_0 \in \mathscr{A}$ with $\Gamma_0(y_0) = a_0$, hence $\Gamma_0(\exp(2\pi i y_0)) = \exp(2\pi i a_0) = \Gamma_0(q_0)$ .
Since $\Gamma_0$ is injective, we get $q_0 = \exp(2\pi i y_0)$.
As in the second part of 6.2 we see, that $m \in \mathscr{O}^{\mathscr{B}*}(E^*) \cap \mathscr{M}^{\mathscr{B}*}(E)$ , defined by
$v_0(\sigma) = p^*m(\sigma)\exp(y_0(\sigma-\sigma_0))$ on H, is actually holomorphic and regular in 0.
Let $x := m(0) \in \mathscr{B}^*$.
ii) We have $\{y_0\}^c \subset \{q_0\}^c$ , that is for $z \in \{q_0\}^c$ we get by Proposition 4.4:

$$p^*h_z = \alpha_{p^*m}\exp(C_{y0}(\sigma-\sigma_0))(z) = p^*\Big(\alpha_{m(\zeta)} \sum_{j=0}^{r} (\zeta/\zeta_0)^{\lambda_j} P_j(z)\Big).$$

with $\lambda_i \in \mathrm{PSp}({}^{C}y_0) \cap \mathbb{Z}$ , where the $P_i$ are the projections of $\{q_0\}^c$ onto the corresponding eigenspaces of ${}^{C}y_0$ .
Obviously $\{y_0\}^c \subset \mathscr{A}$ and $\Gamma_0 \mid \{y_0\}^c = \alpha_x$ .
Now we see from Lemma 3.5, that
$\alpha_x : \{q_0\}^c \to \{\exp(2\pi i a_0)\}^c$
$\alpha_x : \{y_0\}^c \to \{a_0\}^c$

are isomorphisms, i.e. $\mathrm{PSp}({}^{C}y_0) \cap \mathbb{Z} = \mathrm{PSp}({}^{C}a_0) \cap \mathbb{Z}$ and the corresponding eigenspaces resp. projections are related via $\alpha_x$. Since $\Gamma_0$ is injective, we get $\mathrm{PSp}(\, C_{f(0)}\,) \cap \mathbb{N} = \varnothing$.

q.e.d.

Concerning the structure of $\mathscr{A}$ we get :

**Proposition 6.14:** *Suppose there is a fundamental solution* v *of (*) of the form :*

$$v(\sigma) = p^*h(\sigma) \exp(a_0\, \sigma) \text{ on } H \text{ with } h \in \mathscr{O}^{\mathscr{B}^*}(E) \text{ and } h(0) = \mathbf{e}\,.$$

*Let* $x \in \mathscr{B}^*_0$ *be defined by* $v = v_0\, x$ *and set* $y_0 := \alpha_x(a_0)$.

*Then :* $q_0 = \exp(2\pi i y_0)$ , $\exp(2\pi i a_0) = \alpha_x^{-1}(\, q_0)$ ,

$\{ y_0\}^c \subset \mathscr{A}$ *and* $\Gamma_0 = \alpha_x^{-1}\, P_0$ *on* $\mathscr{A}$*, with* $P_0$ *the projection of* $\{ q_0\}^c$ *onto* $\{ y_0\}^c$.

$\Gamma_0$ *is clearly surjective (cf. Proposition 6.4).*

*We have* $\mathscr{A} = \{ y_0 \}^c \oplus \mathscr{J}$ *in the sense of vector spaces, where* $\mathscr{J}$ *is spanned by the eigenvectors corresponding to natural eigenvalues of* ${}^{C}y_0$ *; furthermore* $\mathscr{J} = \mathrm{Ker}\ \Gamma_0 = \mathrm{Im}\,({}^{C}y_0 \mid_{\mathscr{A}})$ *is a closed nilpotent two-sided ideal in* $\mathscr{A}$*, thus* $\mathscr{J} \subset \mathrm{rad}(\mathscr{A})$ *.*

The **proof** is essentially analogous to the proof of Proposition 6.13, where we now have to take into account, that $\mathscr{J}$ does not reduce to {0}.

Proposition 6.8 shows, that the statement of 6.14 in general cannot be sharpened.

Due to Proposition 6.1 we have in a finite-dimensional algebra $\mathscr{B}$ always $q_0 \in \exp(\mathscr{A})$ and $\mathscr{A}$ is always a full subalgebra of $\mathscr{B}$. Let us conclude with the matrix-case:

**Proposition 6.15:** *For* $\mathscr{B} = M_n(\mathbb{C})$ *the following statements are equivalent :*

*i)* $\mathrm{PSp}(\, C_{f(0)}\,) \cap \mathbb{N} = \varnothing$ .

*ii)* $\mathrm{PSp}(\, C_{f(0)}\,) \cap \mathbb{Z} = \{0\}$ .

*iii)* $\mathrm{Sp}(\, C_{f(0)}\,) \cap \mathbb{N} = \varnothing$ .

*iv)* $(\mathrm{Sp}(a_0) - \mathrm{Sp}(a_0)\,) \cap \mathbb{Z} = \{0\}$ .

*v)* $\mathscr{A} = \{q_0\}^c$ *and* $\Gamma_0 : \mathscr{A} \to \{a_0\}^c$ *is bijective.*

*vi)* $\Gamma_0 : \mathscr{A} \to \{a_0\}^c$ *is bijective.*

*vii)* $\mathscr{A} = \{q_0\}^c$ *and there are* $y_0 \in \mathscr{A}$ *and* $m \in \mathscr{O}^{\mathscr{B}^*}(E)$ *with*

$$v_0(\sigma) = p^*m(\sigma) \exp(y_0(\, \sigma - \sigma_0)) \text{ on } H.$$

*If one of these conditions is satisfied, then :*

*viii) There is a fundamental solution* v *of (*) of the form :*

$$v(\sigma) = p^*h(\sigma) \exp(a_0\, \sigma) \text{ on } H \text{ with } h \in \mathscr{O}^{\mathscr{B}^*}(E) \text{ and } h(0) = \mathbf{e}\,.$$

**Proof:**
a) The equivalence of i, ii, iii und iv is essentially known (cf. [Hi] ), because $\mathcal{B} = M_n(\mathbb{C})$ is semi-prime and left and right multiplication with $a_0$ possess a pure point spectrum due to finite dimension.
b) vii ⇒ viii: Proposition 6.4
c) iii ⇔ v: Propositions 5.1, 6.6 , 6.13 and a .
d) v ⇒ vii: Proposition 6.13.
e) vii⇒ iii: Clear from the proof of 6.13.
f) v ⇒ vi: trivial.
vi ⇒ v: Proposition 6.13. and a. q.e.d.

**Example 6.16:** The condition viii is not equivalent to the others in 6.15, as the following example in $\mathcal{B} = M_2(\mathbb{C})$ shows:

Let $f(\zeta) = \begin{pmatrix} 2 & 0 \\ 2\zeta - 3 & 5 \end{pmatrix}$ , i.e. $a_0 = \begin{pmatrix} 2 & 0 \\ -3 & 5 \end{pmatrix}$ ,

then there is a fundamental solution of the form:
$v = p^*m \exp(a_0\sigma)$ with $m(0) = \mathbf{e}$ .
We can take any

$m(\zeta) = \begin{pmatrix} 1 & 0 \\ \alpha\zeta^3 - \zeta & 1 \end{pmatrix}$ , with $\alpha \in \mathbb{C}$ .

Here we have $q_0 = \mathbf{e}$ and $\{a_0\}^c$ is isomorphic to $\mathbb{C}^2$ . For $\sigma_0 = -1$ $\mathcal{A}$ is the algebra of lower triangular matrices in $M_2(\mathbb{C})$ , $\mathcal{A} = \mathcal{A}^{cc}$ .
$q_0$ and $\exp(2\pi i a_0)$ are obviously conjugate, but f is not k-invariant for any k.

**Proposition 6.17:** *Let* $\mathcal{B} = M_n(\mathbb{C})$ .
*If* $\mathrm{PSp}( C_{f(0)} ) \cap \mathbb{N} = \emptyset$ *or if* $q_0^m = \mathbf{e}$ *for some* $m \in \mathbb{N}$ *then there are* $y_0 \in \mathcal{A}$ *and* $m \in \mathcal{O}^{\mathcal{B}^*}(E)$ *with* $v_0(\sigma) = p^*m(\sigma) \exp(y_0( \sigma-\sigma_0))$ *on* H.
$\Gamma_0 : \mathcal{A} \to \{a_0\}^c$ *is surjective.*

**Proof:** Hille shows ([Hi], p. 234 ff) that (*) has always a Frobenius solution. The assertion then follows from Propositions 2.2 and 6.4 . q.e.d.

Example 6.16 is covered by Proposition 6.17, but usually $q_0$ is not known. The example of Gantmacher ([Ga], Vol.2, p.145f) shows that the condition $\exp(2\pi i m a_0) = \mathbf{e}$ for some $m \in \mathbb{N}$ is not sufficient for the assertion of 6.17.

Independently of Proposition 6.1 we may consider the matrix case as special case of:

**Proposition 6.18:** *Suppose that (*) admits of an Frobenius solution* $v(\sigma) = u(\sigma) \exp(a_0\sigma)$ . *Then there is a* $x \in \mathcal{B}^*_0$ *with* $\mathrm{Sp}(x) = \{1\}$, $\lim \exp(\sigma\, C_{a_0})(x) = \mathbf{e}$ for $\sigma \to -\infty$ *and* $p^*g_0(\sigma) = \alpha_{u(\sigma)} \exp(\sigma\, C_{a_0})(x \exp(2\pi i a_0))$ .

**Proof:** Let $v(\sigma) = u(\sigma) \exp(a_0\sigma)$ be a Frobenius solution of (*) , where u is of the form described in Proposition 2.2. Then equally $w(\sigma) = u(\sigma+2\pi i) \exp(a_0\sigma)$ is a Frobenius solution of (*), so there is $x \in \mathcal{B}^*$ with $u(\sigma+2\pi i) \exp(a_0\sigma) = u(\sigma) \exp(a_0\sigma)\, x$ .
Since $p^*g_0 = u(\sigma+2\pi i) \exp(2\pi i a_0)\, u(\sigma)^{-1}$ we get
$p^*g_0 = \alpha_u \exp(\sigma\, C_{a_0})(x \exp(2\pi i a_0))$ and hence
$\lim \exp(\sigma\, C_{a_0})(x) = \mathbf{e}$ for $\sigma \to -\infty$ . Consequently $x \in \mathcal{B}^*_0$ and $\mathrm{Sp}(x) = \{1\}$ .
q.e.d.

**Corollary 6.19:**
*If* $\mathcal{B}$ *contains no quasi-nilpotent elements then any Frobenius solution of (*) is of type 1.*

**Proof:** Apply Proposition 2.2 or independently Proposition 6.18. q.e.d.

Let us note the following extension of 5.4 vi:

**Proposition 6.20:** *Suppose there is a* $m \in \mathbb{N}$ *with* $\exp(2\pi i a_0)^m = \mathbf{e}$ . *Then*

*i) There is a Frobenius solution of (*).*
*ii)* $q_0 \in \exp(\mathcal{A})$ *(cf. 6.4).*
*iii) There is a unique minimal* $n \in \mathbb{N}$ *with* $(g_0(\zeta)^m - \mathbf{e})^n = 0$ *for all* $\zeta \in E$.

**Proof:** i) $C_{a_0}$ belongs to a finite dimensional subalgebra of $BL(\mathcal{B})$ . Apply Hille's result cited above.
ii) $a_0$ belongs to a finite dimensional subalgebra of $\mathcal{B}$ .
iii) Clear from 5.1. q.e.d.

## 7. Proof of Theorem 1.2

Whereas the proof of Theorem 1.1 of the introduction - in so far it concerns the *only if* part - is a mere corollary of Proposition 5.2, the proof of Theorem 1.2 requires quite a lot of the results obtained in this paper and of course some intrinsic properties of the example to be found.

Let us restate it:

**Theorem 1.2:**
*There are Banach algebras* $\mathscr{B}$ *with* $\exp(\mathscr{B})$ *not open in* $\mathscr{B}^*_0$ *and in* $\mathscr{B}$ *a Fuchsian differential equation (*)(over* $\mathbb{P}_1$*), such that the factor of automorphy of a fundamental solution at at least one singularity does not belong to* $\exp(\mathscr{B})$ .

We have to show, that there is a Fuchsian differential equation ( i.e. all singularities are Fuchsian ) over $\mathbb{P}_1$ , such that locally around at least one singularity the factor of automorphy of a fundamental solution does not belong to $\exp(\mathscr{B})$ , where additionally $\mathscr{B}$ has to be specified.
As indicated in the introduction, we take $\mathscr{B} = BL(\mathscr{X})$, the Banach algebra of bounded linear operators on $\mathscr{X} := C[0,1]$, the Banach algebra of continuous complex-valued functions on the real interval [0,1], taken with the norm of uniform convergence.
Let $A_0 \in \mathscr{B}$ be defined by $A_0f(t) := tf(t)$, $f \in \mathscr{X}, t \in [0,1]$, and
$N_0 \in \mathscr{B}$ , be defined by $N_0(f)(t) := f(1-t)$ , $f \in \mathscr{X}, t \in [0,1]$ and consider the differential equation :

$$(+) \qquad \zeta w'(\zeta) = ( A_0 + N_0\, \zeta/(1-\zeta) )\, w(\zeta) , \zeta \in E^* = \{ \zeta \in \mathbb{C} : 0 < |\zeta| < 1\} .$$

We have $A_0N_0 - N_0A_0 = (2A_0-\mathbf{id})N_0$ , this differential equation is a hypergeometric equation over $\mathbb{P}_1$.
As in example 5.7 we see $\mathfrak{f}^{\,c} = \{A_0\}^c \cap \{N_0\}^c$ .
If $\tau \in \mathscr{X}$ denotes the identity on [0,1] , then $Sp(\tau) = [0,1]$ , $A_0 = L_\tau$ .
Thus $Sp(A_0) = [0,1]$ , $A_0$ has no eigenvalues at all and so does $L_{A0}$ , especially $L_{A0}$ is injective.
The Approximation Theorem of Weierstraß tells us : $\{A_0\}^c = \{L_g : g \in \mathscr{X}\}$ , hence $\{A_0\}^c$ is commutative, semi-simple and contains no non-trivial idempotents.
In consequence of $N_0^2 = \mathbf{id}$, Lemma 3.8 (applied to $n = N_0$ and $a = A' := A_0 - \mathbf{id}/2$) provides :
$\{A_0\}^c \cap \{N_0\}^c = \{A'^{\,2}\}^{cc} = \mathfrak{f}^{\,c}$ , $\mathfrak{f}^{\,cc} = \{A'^{\,2}\}^c$ , $PSp(C_{A0} \mid \{A'^{\,2}\}^c ) = \{0\}$ .

If now $Q_0$ is the factor of automorphy of a normalized fundamental solution V of (+) on H, then by Proposition 5.6 we have :
$\{A'^{\,2}\}^{cc} = \mathfrak{F}^{\,c} \subset \{A_0\}^c \cap \{Q_0\}^c \subset \{Q_0\}^c$ .
It is true that remark 5.12 allows us to consider V as a function with values in $\{A'^2\}^c$ , but to decide whether $Q_0$ belongs to $\exp(\mathscr{B})$ , the restriction of our considerations to $\{A'^{\,2}\}^c$ is at best justified if indeed: $\{Q_0\}^c \subset \{A'^{\,2}\}^c$.
Suppose now we had $\{Q_0\}^c$ commutative, then $\{Q_0\}^c \subset \{A'^{\,2}\}^c$ .
Assume $Q_0 \in \exp(\mathscr{B})$ , then by $\{Q_0\}^c \subset \{A'^{\,2}\}^c$, hence $Q_0 \in \exp(\{A'^{\,2}\}^c)$, and by applying propositions 6.12 and 6.4 to $\{A'^{\,2}\}^c$ , we get a fundamental solution W of (+) on H of the form

$$W(\sigma) = p^*G(\sigma) \exp(A_0\, \sigma) \text{ on } H ,$$

where G is $\{A'^{\,2}\}^c$- valued, holomorphic and regular on E with $G(0) = \mathbf{id}$.
A simple calculation shows for $T := G'(0) \in \{A'^{\,2}\}^c$: $C_{A0}(T) = T - N_0$ .
Again by Lemma 3.8 and the structure of $\{A_0\}^c = \{A'\}^c$we get a non-vanishing $f \in \mathscr{X}$ with $C_{A0}(T) = L_f N_0$ . That means $T = L_{1+f}N_0$ , $L_f N_0 = L_{(2\tau-1)\,(1+f)}N_0$ , $f = (2\tau-1)\,(1+f)$ , $f = (2\tau-1)/(2(1-\tau))$. But f is obviously not continuous on [0,1] .

Thus $Q_0$ does not belong to $\exp(\mathscr{B})$ , by Proposition 5.2 $\exp(2\pi i A_0)$ is not an interior point of $\exp(\mathscr{B})$ .

It remains to show that $\{Q_0\}^c$ is commutative, and again by Proposition 5.1 it will be sufficient to prove, that $C_{A0}$ has no integer eigenvalues different from 0 (relative to $\mathscr{B}$) .

**Lemma 7.1:**
*i)* $Sp( C_{A0} ) = [ -1,1]$ .
*ii)* $PSp( C_{A0} ) \cap \mathbb{Z} = \{0\}$
*iii)* $PSp( C_{A0} ) = (-1,1)$ .

**Proof:**
Let $\mathscr{P}$ be the subalgebra of polynomials in $\tau$ of $\mathscr{X}$, which is dense by the Approximation Theorem of Weierstraß. We denote by 1 equally the constant function 1 on [0,1] .
If $T \in \mathscr{B}$ with $TA_0 = A_0T$, then
$T(\tau) = \tau\, T(1)$ resp. $T(\tau^k) = \tau^k\, T(1)$ by induction for $k \in \mathbb{N}$ .
That means $T(p) = L_{T(1)}\, p$ for $p \in \mathscr{P}$ and thus $\{A_0\}^c = \{L_f : f \in \mathscr{X} \}$.
Since $\tau$ is a generator of $\mathscr{X}$, we have $\{A_0\}^c = V(A_0)$ .
Since $N_0^2 = \mathbf{id}$ we have with $P := ( \mathbf{id} + N_0)/2$ and und $Q := \mathbf{id} - P$ two orthogonal continuous projections onto the eigenspaces of $N_0$ corresponding to the eigenvalues 1 resp. -1 , $N_0 = P - Q$.

i)

Obviously $\| N_0\| = 1$ , $A_0N_0 - N_0A_0 = (2A_0-\mathbf{id})N_0$ and $Sp(C_{A0}) \subset [-1,1]$ .

For $\lambda \in \mathbb{C} \setminus [-2,2]$ with $| \lambda| > \| \mathbf{id} - 2A' \|$ we get from Lemma 3.8 :

$( \lambda \,\mathbf{id} - C_{A0} )^{-1} (N_0) = (\lambda \,\mathbf{id} - 2A')^{-1} N_0 = ((\lambda+1)\, \mathbf{id} - 2A_0)^{-1} N_0$, which by analytic continuation remains valid for all $\lambda \in \mathbb{C} \setminus [-1,1]$ .

Since $\| N_0\| = 1$ and $N_0^2 = \mathbf{id}$ we get:

$\| ( \lambda \,\mathbf{id} - C_{A0} )^{-1}\|_{BL(\mathscr{B})} \geq \| (\lambda \,\mathbf{id} -2A')^{-1}N_0 \|_{\mathscr{B}} \geq \| (\lambda \,\mathbf{id} - 2A')^{-1} \|_{\mathscr{B}} = \| (\lambda 1 +1 -2\tau )^{-1} \|_{\mathscr{X}}$

because the left regular representation of $\mathscr{X}$ is an isometry.

Now for $\lambda \in \mathbb{C} \setminus [-1,1]$ with Rea $\lambda \in [-1,1]$ we get for the right hand side in this inequality (set $t = ( \text{Rea } \lambda + 1) /2$ ) :

$\| (\lambda +1 - 2\tau )^{-1} \| \geq 1/ |\text{Im } \lambda|$.

Thus: $Sp(C_{A0}) = [-1,1]$ .

ii)

For $\mu \in \mathbb{R}$ define $E_\mu : \mathscr{P} \to \mathscr{P}$ by: $E_\mu (p) (t) = p(t-\mu )$ .

We thus get a 1-parameter-group of algebra automorphisms of $\mathscr{P}$; since $A_0 E_\mu - E_\mu A_0 = \mu E_\mu$ the $E_\mu$ are not bounded for $\mu \neq 0$ .

Suppose now we had for $\mu \neq 0$ a $T \in \mathscr{B}$ with $T \neq 0$ and $A_0T - TA_0 = \mu T$, then $\mu \in [-1,1]$.

For $k \geq 0$ let $p_k \in \mathscr{P}$ be defined by $p_k := E_\mu ( \tau^k) = E_\mu ( \tau)^k = (\tau - \mu )^k$ .

We see $\tau T(1) - T(\tau) = \mu T(1)$, i.e. $T(\tau) = p_1 T(1)$ resp. by induction on k:

$T( \tau^k) = p_k T(1)$ resp. $T(p) = E_\mu (p ) T(1)$ for all $p \in \mathscr{P}$ .

T(1) is singular because otherwise we had $E_\mu$ bounded.

Since $N_0A_0N_0 = \mathbf{id} -A_0$ we have $N_0TN_0$ as eigenvector corresponding to $- \mu$.

It is therefore sufficient, to show that 1 is not an eigenvalue of $C_{A0}$ .

Assume the contrary and consider the polynomial $p := 2 \tau - 1$.

Then $\|p^k\| = 1$ for all natural k, but for $t < 1$ we have $E_1 (p) (t) = 2t - 3$, hence $| E_1 (p) (t) | > 1$.

But $|T(p^k)(t)| = | E_1 (p) (t) |^k |T(1)(t)|$, so the boundedness of T requires T(1) to vanish for $t < 1$, but then $T(1) = 0$ , $T =0$ .

iii) Since $0 \in PSp(C_{A_0})$ it suffices to show that (0,1) belongs to the point spectrum of $C_{A_0}$ .

Let $r \in (0,1)$ and take an arbitrary non-zero function h from C[0,1] with $h(t) = 0$ for $t < r$ .

Obviously $h(r) = 0$ . For $f \in C[0,1]$ set $T(f)(t) := 0$ for $t < r$ resp. $T(f)(t) = f(t-r)h(t)$ for $t > r$ .

Then T(f) is continuous on [0,1], $T: C[0,1] \to C[0,1]$ is linear.

$|T(f)(t)| \leq \|h\| \sup\{|f(t)| : t \in [0,1-r] \}$ , so that T is bounded.

Since $T(1) = h$ and $A_0T-TA_0 (p) = r\, T(p)$ for all polynomials p , T is indeed an eigenvector of $C_{A_0}$ corresponding to the eigenvalue r . q.e.d.

With 7.1 ii the proof of Theorem 1.2 is now complete. In view of Proposition 5.11 with the isometric pseudo-involution ~ on $\mathcal{B}$ (cf. section 3), we see that for $\sigma_0$ real: $\{Q_0\}^c$ is commutative, semi-simple and symmetric, $\mathcal{A} = \{Q_0\}^c$, and the closure of $\Gamma_0(\mathcal{A})$ is a selfadjoint subalgebra of the B*-algebra $\{A\}^c$, that is: $\Gamma_0$ is equivalent to the Gelfand-representation of $\mathcal{A}$.

**Corollary 7.2:** *Let* $[a,b]$ *be a non-void compact real interval and consider the identity* $\tau$ *on* $[a,b]$ *as element of* $C[a,b]$ *and* $A := L_\tau$ *as element of* $\mathcal{B} = BL(C[a,b])$.

*i) If* $b-a < 1$ *then* $\exp(2\pi i A)$ *is an interior point of* $\exp(\mathcal{B})$ .

*ii) In general* $\exp(2\pi i A/(b-a))$ *lies on the boundary of* $\exp(\mathcal{B})$ *relative to* $\mathcal{B}^*_0$.

*iii)* $Sp(C_A) = [-(b-a), b-a]$ , $PSp(C_A) = ( -(b-a), b-a)$ .

**Proof:**

i) If $b - a < 1$ then $Sp(C_A) \subset (-1,1)$ . Apply Proposition 4.1 .

ii) The homeomorphism $\Phi : [0,1] \to [a,b]$ , $\Phi(t) := (b-a)\, t + a$ induces an isometric algebra isomorphism $\Phi^* : C[a,b] \to C[0,1]$ , $\Phi^*(f)(t) = f(\Phi(t))$, and consequently an isometric algebra isomorphism $\Phi_* : BL(C[0,1]) \to BL(C[a,b])$ , $\Phi_*(T) = \Phi^{*-1} \circ T \circ \Phi^*$ , $T \in BL(C[0,1])$ , so that $\Phi_*(A_0) := ( A - a\,\mathbf{id})/(b-a)$ , where $A_0$ , $A$ stands for the left multiplication with the identity on the resp. interval.

Note that $\Phi_*(N_0)\,(f)(s) = f(b + a - s)$, $s \in [a,b]$, $f \in C[a,b]$ . So we can apply always Lemma 3.8 to $a = A - \frac{1}{2}(a+b)\,\mathbf{id}$ and $n = \Phi_*(N_0)$ .

iii) clear. q.e.d.

We now consider the case $\mathcal{X} = C[-1,1]$, $\mathcal{B} = BL(C[-1,1])$ with $A \in \mathcal{B}$ defined as in 7.2 and $N \in \mathcal{B}$ defined by $N(f)(t) = f(-t)$ , $f \in \mathcal{X}$, $t \in [-1,1]$ . We note that $N_0 = \Phi_*^{-1}(N)$, $\Phi_*^{-1}(A) = 2A_0 - \mathbf{id}$ .

**Lemma 7.3:**

*i)* $Sp( C_A ) = [ -2,2]$ .

*ii)* $PSp( C_A ) \cap \mathbb{Z} = \{-1, 0, 1\}$.

*iii)* $PSp(C_{N-A} \mid_{\{A^2\}^c} ) = \{0\}$ , $\{N-A\}^c \subset \{A^2\}^c$ *contains non-trivial idempotents .*

*iv)* $\rho(C_{N-A}) \leq 2\sqrt{2}$ .

*v) Let* $K \subset [-1,1]$ *be a finite set symmetric to 0 , that is* $K = -K$,
$\mathcal{J} := \{ f \in \mathcal{X} : f|_K = 0 \}$ *and* $\mathcal{C} := \{ T \in BL(\mathcal{X}) : T(\mathcal{J}) \subset \mathcal{J} \}$ . *Then* $\mathcal{C}$ *is a closed subalgebra of* $\mathcal{B}$ *with unit* **id** ; *if* $0 \notin K$ *then* $\{A^2\}^c \subset \mathcal{C}$ .

*vi) If* $n := \#K$ *and* $K = \{ t_1, \dots, t_n \}$ *with* $t_i < t_{i+1}$, $i = 1, \dots, n-1$ , *then there is a continuous algebra homomorphism* $\Psi\colon \mathscr{C} \to M_n(\mathbb{C})$ *with* :
$\Psi(A) = \mathrm{diag}(t_1, \dots, t_n)$ *is a diagonal matrix and* $\Psi(N)$ *is the matrix with one's on the diagonal from the upper right to the lower left corner, all other entries vanish.*
*For* $n=2$ *we have* $\Psi\colon \{A^2\}^c \to M_2(\mathbb{C})$ *surjective , thus* $\Psi$ *is irreducible .*

*vii) If* $<A,N>$ *denotes the closed subalgebra of* $\mathscr{B}$, *generated by* A, N *and* **id** , *then* $<A,N>$ ,$\{A^2\}^c$ , $\mathscr{B}$ *are all semi-simple . There is a family of two-dimensional irreducible representations of* $\{A^2\}^c$ *separating the points of* $\{A^2\}^c$ .

*viii) For even* $n$ *:* $\{ T \in \{A^2\}^c : \Psi(T) = 0 \} = \{ T \in \{A^2\}^c : T(1) , T(\tau) \in \mathscr{J} \}$ ,
*i.e.* $\mathrm{codim}_{\mathbb{C}} \mathrm{Ker}\Psi = 2n$, *for* $n>2$ $\Psi$ *is no longer surjective .*

**Proof:**

i + ii are clear from the previous considerations.
In this concrete case Lemma 3.8 ($a=A$, $n=N$) allows some refinements:
Since $N^2 = \mathbf{id}$ we have with $P := ( \mathbf{id} + N)/2$ and und $Q := \mathbf{id} - P$ two orthogonal continuous projections onto the eigenspaces of N corresponding to the eigenvalues 1 resp. -1 , $N = P - Q$.
The Approximation Theorem of Weierstraß yields for $T \in \{A^2\}^c$:

$$TP = L_{T(1)}P , \quad TA = L_fQ + L_{T(\tau)} , \quad AT = L_{\tau T(1)}P + L_{T(\tau)}Q .$$

Thus $AT - TA = L_fN$ with $f = \tau T(1) - T(\tau)$ .

Note that $\{A^2\}^{cc} = V(A^2)$ .

It should be remarked that the algebra $\{A^2\}^c$ is quite "small": the only algebra automorphisms of $\mathscr{X}$ contained in $\{A^2\}^c$ are **id** and N, while the set of all algebra automorphisms forms a discrete, closed uncountable subset of the unit sphere of $\mathscr{B}$, whose elements pairwise have a distance > 1/2 from each other .

iii+iv) Let $X := N - A$, then on $\{A^2\}^c$ we get from $X^2 = \mathbf{id} + A^2$ :

$$(C_X)^k = (-2R_X)^{k-1}C_X \quad \text{for } k > 0 .$$

If $C_X(T) = rT$ with $T \in \{A^2\}^c$ and r different from 0 , then :

$$rT= -2TX , -2TX^2 = rTX = r(-r/2)T = -r^2/2\, T , \text{ i.e. } (\mathbf{id}+A^2)T = r^2/4\, T , r^2/4 \in \mathrm{PSp}(\mathbf{id}+A^2) .$$

Thus $C_X$ - restricted to $\{A^2\}^c$ - has 0 as unique eigenvalue.

Suppose $\{X\}^c$ did not contain non-trivial idempotents, then $\mathrm{Sp}(X)$ is connected, thus $\mathrm{Sp}(X) = [1 , \sqrt{2}]$ or $\mathrm{Sp}(X) = [-\sqrt{2} , - 1]$ , i.e. $\mathrm{Sp}(C_X) \subset [ 1-\sqrt{2} , \sqrt{2}-1 ]$,
$\rho(C_X) \le \sqrt{2}-1$ , $\mathrm{Sp}(\alpha_X) \subset [ 1/\sqrt{2} , \sqrt{2} ]$ is purely positive.

But then according to 4.9 : $\{X\}^c = \{X^2\}^c = \{A^2\}^c$, but $\{A^2\}^c$ contains with P,Q as above obviously non-trivial idempotents. Thus $\mathrm{Sp}(X)$ is not connected.
From $\mathrm{Sp}(X) \subset [-\sqrt{2} , - 1] \cup [1 , \sqrt{2}]$ we get $\rho(C_X) \le 2 \sqrt{2}$ .

Analogously to 7.1 i) we get

$Sp(X) = [-\sqrt{2}, -1] \cup [1, \sqrt{2}]$
and $Sp(C_X) = \{0\} \cup 2Sp(X) = \{0\} \cup [-2\sqrt{2}, -2] \cup [2, 2\sqrt{2}]$ .

v) $\mathscr{J}$ is obviously a closed ideal in $\mathscr{X}$, Lemma 3.9 shows that $\mathscr{C}$ is closed subalgebra of $BL(\mathscr{X})$ with unit **id** .
Let $T \in \{A^2\}^c$ , $h := T(1) \in \mathscr{X}$ and $f \in \mathscr{X}$ defined by $AT - TA = L_f N$ .
Then: $AT + TA = L_{2\tau h-f}$ .
Suppose we had a $j \in \mathscr{J}$ , such that $T(j)$ is not contained in $\mathscr{J}$.
From $T = L_h P + TQ$ we see that $TQ(j)$ is not contained in $\mathscr{J}$, but $Q(j)$ is . Thus $k := Q(j)$ is different from 0.
Hence:

$\tau T(k) - T(\tau k) = f\, N(k) = - fk$ and

$\tau T(k) + T(\tau k) = 2\tau hk - fk$

I.e. $2\, T(\tau k) = 2\tau hk$ ,

$T(\tau k) \in \mathscr{J}$.

Since $fk$ belongs to $\mathscr{J}$, we have $\tau T(k) \in \mathscr{J}$, hence $T(k) \in \mathscr{J}$ , for $\tau$ has no zeroes on K by assumption. Contradiction.
vi) The map $\beta : \mathscr{X} \to \mathbb{C}^n$ , defined by $\beta(f) = (f(t_1), ..., f(t_n))$ for $f \in \mathscr{X}$, is a surjective algebra homomorphism with $\mathscr{J} = \mathrm{Ker}\, \beta = \{ f \in \mathscr{X}: f|_K = 0 \}$, if $\mathbb{C}^n$ is considered with componentwise operations.
$\beta$ is continuous with $\|\beta\| = 1$.
$\beta$ induces an isometric (*-) isomorphism $\beta' : \mathscr{X}/\mathscr{J} \to \mathbb{C}^n$ (cf. [Ne], $\mathbb{C}^n$ , $\mathscr{X}$ are B*-algebras, $\mathscr{J}$ is selfadjoint and $\beta$ is clearly a *-homomorphism).

By definition of K we have $A,N \in \mathscr{C}$ and $\{A\}^c \subset \mathscr{C}$, thus $\mathscr{C}^c \subset \{A\}^c \subset \mathscr{C}$,
$Z(\mathscr{C}) = \mathscr{C}^c \subset \{A\}^c \cap \{N\}^c$ , $\{A^2\}^c \subset \mathscr{C}^{cc}$.

(All homeomorphisms of [-1,1] which leave K fixed, induce algebra automorphisms of $\mathscr{X}$, which leave $\mathscr{J}$ invariant; thus $\mathscr{C}$ is different from $\{A^2\}^c$ .)

If $T \in \mathscr{C}$, then $T^{\#}(f+\mathscr{J}) := T(f) + \mathscr{J}$ defines a well-defined linear map $T^{\#} : \mathscr{X}/\mathscr{J} \to \mathscr{X}/\mathscr{J}$.
With $T' := \beta'^{-1} \circ T^{\#} \circ \beta'$ we get by $\Psi(T) := T'$ an algebra homomorphism $\Psi: \mathscr{C} \to M_n(\mathbb{C})$ .

We have $\Psi(A) = \mathrm{diag}(t_1, .... t_n)$, $\mathrm{Trace}(\Psi(A)) = 0$ and $\Psi(N)$ is the matrix with 1's on the subordinate diagonal, where the other entries vanish.
Thus : n is even if and only if 0 does not belong to K and this in turn is equivalent to $\mathrm{Trace}(\Psi(N)) = 0$ .

For all $T \in \mathscr{C}$ we have $\Psi(T) \circ \beta = \beta \circ T$ ; since $\beta$ is surjective: $\mathrm{Ker}\,\Psi = \{ T \in \mathscr{C} : \beta \circ T = 0 \}$, $L(\mathscr{J}) \subset \mathrm{Ker}\,\Psi$ . But the linear map $s : \mathscr{C} \to BL(\mathscr{X}, \mathscr{X}/\mathscr{J})$ , defined by $s(T) := \beta \circ T$ , is continuous, thus $\mathrm{Ker}\,\Psi = \mathrm{Ker}\, s$ is a closed ideal in $\mathscr{C}$.

Now $\mathrm{Im}\,\Psi$ is finite-dimensional, so $\Psi$ is continuous.

Let $T = L_f \in \mathrm{Ker}\,\Psi \cap \{A\}^c$ ; from $\beta \circ T = 0$ we get $f \in \mathscr{J}$, thus $L(\mathscr{J}) = \mathrm{Ker}\,\Psi \cap \{A\}^c$ .

Since $\Psi(A)$ is a diagonal matrix with distinct entries, $\{\Psi(A)\}^c$ contains exactly the $n \times n$-diagonal matrices , the set of which we denote by $\Delta$ .

Thus $\{\mathrm{Im}\Psi\}^c \subset \Delta$ . More precisely: $\{\mathrm{Im}\Psi\}^c$ contains only diagonal matrices of the form $\mathrm{diag}(s_1,\ldots,s_n)$ with $s_1 = s_n$ , $s_2 = s_{n-1}$ , etc. Hence for $n = 2$ : $\{\mathrm{Im}\Psi\}^c = \mathbb{C}\,\mathbf{id}$ .

But for $n=2$ the matrices $\Psi(A)$ , $\Psi(N)$ , $\Psi(AN)$ and $\Psi(\mathbf{id})$ are linearly independent over $\mathbb{C}$, what implies , that $\Psi$ is surjective und thus irreducible (and hence continuous).

vii) The closed subalgebra of $\mathscr{B}$, generated by A, N and **id** , lies in $\{A^2\}^c$ ; therefore the restriction $\Psi: \langle A,N \rangle \to M_2(\mathbb{C})$ is surjective and hence irreducible too.

For $\alpha \in (0,1]$ and $K_\alpha := \{-\alpha, \alpha\}$ we get so a family of irreducible representations $\Psi_\alpha: \langle A,N \rangle$ resp. $\{A^2\}^c \to M_2(\mathbb{C})$ . The construction shows $\bigcap \mathrm{Ker}\,\Psi_\alpha = \{0\}$ , i.e. $\langle A,N \rangle$ resp. $\{A^2\}^c$ are semi-simple.

Note that to achieve $\bigcap \mathrm{Ker}\,\Psi_\alpha = \{0\}$ it is sufficient to vary $\alpha$ in a dense subset of $(0,1]$ .

( To see that $\mathscr{B}$ is itself semi-simple, it is easy to show that $L := \{ T \in \mathscr{B}: T(1) = 0\}$ is a maximal modular left-ideal of $\mathscr{B}$ with $[\, L : \mathscr{B}\,] = \{ T \in \mathscr{B} : T\mathscr{B} \subset L \} = \{0\}$ . This argument applies to the operator algebra of any commutative, semi-simple unital Banach algebra.)

viii) Let $T \in \{A^2\}^c$ with $\Psi(T) = 0$ .

If $g \in \mathscr{X}$ is defined by $AT - TA = L_g N$ , then $g \in \mathscr{J}$.

From $AT + TA = L_f$ with a suitable $f \in \mathscr{X}$ we see likewise: $f \in \mathscr{J}$.

But $\tau T(1) + T(\tau) = f$ and $g = \tau T(1) - T(\tau)$ , hence $T(1)$ ,$T(\tau) \in \mathscr{J}$ .

Conversely , if $f,g \in \mathscr{J}$ , we get $\Psi(AT) = \Psi(TA) = 0$, hence:

$\beta \circ AT = \beta \circ TA = 0$. Thus $\mathrm{Im}\, AT \subset \mathscr{J}$, $\beta \circ T = 0$ , $\Psi(T) = 0$ .

q.e.d.

Now let us consider the differential equation :

(+) $\quad \zeta w'(\zeta) = ( A + N\, \zeta/(1-\zeta) )\, w(\zeta)$ , $\zeta \in E^* = \{ \zeta \in \mathbb{C} : 0 < |\zeta| < 1\}$ .

Lemma 3.8 provides :
$\{A\}^c \cap \{N\}^c = \{A^2\}^{cc} = \mathfrak{J}^c$ , $\mathfrak{J}^{cc} = \{A^2\}^c$ , $PSp(C_A \mid \{A^2\}^c) = \{0\}$ .

Proposition 4.1 then tells us : $\{A^2\}^c \cap \{\exp(2\pi iA)\}^c = \{A\}^c$.
If now $Q_0$ is the factor of automorphy of a normalized fundamental solution V of (+) on H, then by Proposition 5.6 we have : $\{A^2\}^{cc} = \mathfrak{J}^c \subset \{A\}^c \cap \{Q_0\}^c \subset \{Q_0\}^c$ .
It is easy to see, that $Q_0$ cannot have a logarithm in $\{A^2\}^c$, especially $\{Q_0\}^c$ is not commutative.
Because otherwise (+) has a fundamental solution of the form $v(\sigma) = p^*h(\sigma)\exp(A\sigma)$ with $h \in \mathscr{O}^{\mathscr{B}^*}(E)$ and $h(0) = \mathbf{id}$ , since this holds for the equation (+) transported to $M_2(\mathbb{C})$ via $\Psi_\alpha$ for $\alpha \in (0,1)\backslash\{1/2\}$ . But then considering $h'(0)$ produces a contradiction as above.
A Frobenius solution of type $n > 1$ is however not excluded.
But we may use the representations $\Psi_\alpha$ to show that (+) has no Frobenius solution in $\{A^2\}^c$.

For $\mathscr{A} \subset \{Q_0\}^c$ we have $\text{rad}(\mathscr{A}) = \text{Ker}\,\Gamma_0$ ; by Propositions 6.2 and 6.4 $Q_0$ has no logarithm in $\mathscr{A}$, which in turn contains no non-trivial idempotents because these were nilpotent.
If we denote by $\mathscr{L}$ the set of single-valued solutions of (**) in E* and by $o(h)$ the order of $h \in \mathscr{L}$ in 0, then by 5.1 we have $o(h) \in \{-1,0,1\}$ for all $h \in \mathscr{L}$ different from zero.
So $(\text{rad}(\mathscr{A}))^2 = \{0\}$ .

Clearly $\mathscr{A}' := \mathscr{A}/\text{rad}(\mathscr{A})$ is commutative and semi-simple. If we suppose the base point $\sigma_0 \in H$ with $V(\sigma_0) = \mathbf{id}$ to be real, then $\mathscr{A}'$ is symmetric with respect to ~ and the map $\Gamma : \mathscr{A}' \to (\Gamma_0(\mathscr{A}))^- \subset \{A\}^c$ induced by $\Gamma_0$ is equivalent to the Gelfand representation of $\mathscr{A}'$.
If $\Gamma_0 : \mathscr{A} \to \{A\}^c$ was surjective, then we would have $Q_0 \in \exp(\mathscr{A})$ by proposition 6.1.
Thus $\Gamma_0$ is not surjective. If $(\Gamma_0(\mathscr{A}))^- = \{A\}^c$ then the image of $Q_0$ under the canonical projection would have a logarithm in $\mathscr{A}'$ . But then, since $\text{rad}(\mathscr{A}) = \text{Ker}\,\Gamma_0$ , the same argument shows $Q_0 \in \exp(\mathscr{A})$ . Hence $(\Gamma_0(\mathscr{A}))^-$ is isometrically *-isomorphic to a closed, <u>strict</u> unital selfadjoint, hence full subalgebra of $\mathscr{X}$ containing $\exp(\pm 2\pi i\tau)$ .
Since $\{A\}^c$ is commutative, we get $\{A\}^c \subset \Gamma_0(\mathscr{A})^c \subset \{A^2\}^c \cap \{\exp(2\pi iA)\}^c = \{A\}^c$,
so we get a strict inclusion $(\Gamma_0(\mathscr{A}))^- \subset \Gamma_0(\mathscr{A})^c = \Gamma_0(\mathscr{A})^{cc} = \{A\}^c$ .
We note $\mathscr{A}^c \subset \{A^2\}^c$ , since $\mathfrak{J}^c \subset \mathscr{A}$ .
For the center $Z(\mathscr{A})$ of $\mathscr{A}$ we get $Q_0 \in Z(\mathscr{A}) \subset \{A^2\}^c$ , $\mathfrak{J}^c \subset Z(\mathscr{A})^c \subset \{Q_0\}^c$ , thus $\{Q_0\}^{cc} \subset \{A^2\}^c$ . But then a solution $h_z$ of (**) for $z \in \{Q_0\}^{cc}$ has to be holomorphic 0, hence $\{Q_0\}^{cc} \subset \mathscr{A}$ , even $\{Q_0\}^{cc} \subset Z(\mathscr{A})$ .
Analogously we get $\{Q_0\}^c \cap \{A^2\}^c \subset \mathscr{A}$.
From $\mathscr{A}^c \subset \{Q_0\}^c \cap \{A^2\}^c \subset \mathscr{A}$ it follows: $Z(\mathscr{A}) = \mathscr{A}^c$ .

Suppose now we had $Q_0 \in \exp(\mathcal{B})$ , that is there is a $Y \in \{Q_0\}^c$ with $Q_0 = \exp(2\pi i Y)$ . The corresponding solution $h_Y$ of (**) has a pole of first order in 0, so there is a function $h \in \mathcal{O}^{\mathcal{B}*}(E)$ with $h(\zeta) = \zeta h_Y(\zeta)$ , $C_A(T) = - T$ with $T := h(0)$, $T^2 = 0$ .

$C_Y$ must have non-trivial integer eigenvalues, because otherwise $Y \in \{Q_0\}^{cc} \subset \mathcal{A}$.

Note that in contrast to Proposition 4.7 $\{Y\}^c$ is not contained in $\mathcal{A}$.

Without loss of generality we may assume that Y is regular, so there are $k \in \mathbb{Z}$ and $g \in \mathcal{O}^{\mathcal{B}*}(E)$ with $h_Y^{-1}(\zeta) = \zeta^k g(\zeta)$ , $g(0) \neq 0$, $k \in \{-1, 0, 1\}$, g and h commute pointwise.

Hence $\zeta\, \mathbf{id} = \zeta^k g(\zeta)\, h(\zeta)$ , so $k \in \{-1, 0\}$.

Since $Sp(Q_0) = S^1$ we have $Sp(Y)$ real, so if $Y^{-1} \in \mathcal{A}$ then 0 does not belong to $Sp_{\mathcal{A}}(Y^{-1})$ , hence $Y \in \mathcal{A}$. So we get $k = - 1$, $\zeta^2\, \mathbf{id} = g(\zeta)\, h(\zeta)$ .

Let $m \in \mathcal{O}^{\mathcal{B}*}(E^*)$ be defined by $V(\sigma) = m(e^{\sigma}) \exp((\sigma - \sigma_0)Y)$ then it is easy to see that m satisfies the differential equation:

$\zeta^2 m'(\zeta) = ( \zeta f(\zeta) - h(\zeta) )\, m(\zeta)$ with $m(\zeta_0) = \mathbf{id}$ where $\exp(\sigma_0) = \zeta_0$.

Let $n \in \mathbb{Z}$ be the order of $\alpha_m$ in 0 (remember that $\alpha_m$ has at most a pole in 0 since (+) is Fuchsian) , so there is $M \in \mathcal{O}^{BL(\mathcal{B})*}(E^*)$ with $M(0) \neq 0$ and $\alpha_{m(\zeta)} = \zeta^n M(\zeta)$ .

We must have $n < 0$ for otherwise we had $\{Y\}^c \subset \mathcal{A}$ .

So it would require further investigation to see whether such an Y exists. At least we are sure that it does not belong to $\{A^2\}^c$.

Let us briefly consider the differential equation (+) at the other singularities 1 and ∞ :

(++) $\zeta w'(\zeta) = ( -N - A\, \zeta/(1-\zeta) )\, w(\zeta)$ , $\zeta \in E$ ,

resp.

(+++) $\zeta w'(\zeta) = ( (N-A) + N\, \zeta/(1-\zeta) )\, w(\zeta)$ , $\zeta \in E$ .

For (++) we have as for (+):

$\mathfrak{J}^{\,c} = V(A^2) = \{A^2\}^{cc}$ , $\mathfrak{J}^{\,cc} = \{A^2\}^c$ .

Since $\exp(- 2\pi i N) = \mathbf{id}$ , Proposition 5.2 says (in our usual notation) $q_0 \in \exp(\mathcal{B})$ , and Proposition 5.1 provides a nilpotent $M \in \mathcal{A} \subset \{q_0\}^c$ with $q_0 = \mathbf{id} + 2\pi i M$.

But due to the discrete spectrum of $C_N$ we can apply to (++) the method of Frobenius ([Hi], p. 239) ; thus there are $h,g \in \mathcal{O}^{\mathcal{B}}(E)$ with $h(0) = \mathbf{id}$ and a fundamental solution v of (++) on H of the form :

$$v(\sigma) = ( p^*h(\sigma) + \sigma\, p^*g(\sigma) ) \exp(-N \sigma).$$

If q is defined by $(\mathfrak{g}_0^{-1})^* v = v\, q$ then

$$v(\sigma)q = v(\sigma) + 2\pi i p^*g(\sigma) \quad \exp(-N\,\sigma)$$

resp. $\quad q = \mathbf{id} + 2\pi i\, v(\sigma)^{-1} p^*g(\sigma) \quad \exp(-N\,\sigma)$

resp. $\quad 2\pi i p^*g(\sigma) = v(\sigma)\,(q - \mathbf{id})\exp(N\,\sigma)$.

Applying $(\mathfrak{g}_0^{-1})^*$ again, we get:

$$q\,(q - \mathbf{id}) = (q - \mathbf{id})\,, \quad (q - \mathbf{id})^2 = 0\,.$$

Since q and $q_0$ are conjugate, we see $M^2 = 0$ and $q_0 = \exp(2\pi i\, M)$ .

Hence there is $m \in \mathscr{O}^{\mathscr{B}*}(E^*)$ with

$$v_0(\sigma) = p^*m(\sigma)\exp(M(\sigma - \sigma_0)) = p^*m(\sigma)\,(\,\mathbf{id} + M\,(\sigma - \sigma_0)\,) \text{ on } H\,.$$

Some easy calculation shows, that m has in 0 a pole of first order.

For (+++) again we have $\mathfrak{F}^c = \{N-A\}^c \cap \{N\}^c = \{A\}^c \cap \{N\}^c$, and $Sp(N-A) \subset [1, \sqrt{2}] \cup [-\sqrt{2}, -1]$ forces $Sp(\exp(2\pi i\,(N-A)))$ to be contained in a non closed segment of $S^1$. Thus $\exp(2\pi i\,(N-A))$ like $q_0$ are inner points of $\exp(\mathscr{B})$ .

That means that M from above cannot vanish, for otherwise we had
$Sp(\exp(2\pi i\,(N-A))) = S^1 = Sp(\exp(2\pi i A))$ .

Since $\{A^2\}^c$ is a full subalgebra of $\mathscr{B}$, we have likewise $q_0 \in \exp(\{A^2\}^c)$ .

Let $v(\sigma) = p^*m(\sigma)\exp(Y(\sigma - \sigma_0))$ be a normalized fundamental solution of (+++) with values in $\{A^2\}^c$.

For $\alpha \in (0,1]$ and $K_\alpha := \{-\alpha, \alpha\}$ consider the maps (see above): $\Psi_\alpha\colon \{A^2\}^c \to M_2(\mathbb{C})$ .

Now $Sp(\Psi_\alpha(N-A)) = \{\pm\sqrt{(1+\alpha^2)}\}$, so that for fixed $\alpha$ the commutator corresponding to $\Psi_\alpha(N-A)$ has no integer eigenvalues different from 0.

According to Proposition 6.15 $\Psi_\alpha(m)$ is holomorphic and regular in 0 for all $\alpha$ . Since the $\Psi_\alpha$ separate the points in $\{A^2\}^c$, this holds for m as well.

Finally let us indicate, that differential equations such as (+) do not live on their own, but may well be related to more concrete problems.

Suppose we are given a point $\zeta_0 \in E^*$ and a continuous complex-valued function $f_0$ on [-1,1].

Does there exist a complex-valued function $u(\zeta, t)$, defined and continuous on $E^* \times [-1,1]$ such that:

i) For fixed $t_1 \in [-1,1]$ $v(\zeta) := u(\zeta, t_1)$ is complex differentiable at $\zeta \in E^*$ .

ii) For fixed $\zeta_1 \in E^*$ $g(t) := u(\zeta_1, t)$ is continuous on $[-1,1]$ .

iii) We have $u(\zeta_0, t) = f_0(t)$ for $t \in [-1,1]$ .

iv) u satisfies the "differential equation" on $E^* \times [-1,1]$ :

(a) $\qquad \zeta\, \partial/\partial\zeta\; u(\zeta, t) = t\; u(\zeta, t) + \zeta/(1-\zeta)\; u(\zeta, -t)$ ?

Because of the factor $\zeta$ on the left side it seems natural to consider the lifting ( by $\zeta = \exp(\sigma)$) to the left halfplane H, thus:

(b) $\partial/\partial\sigma\ u(\sigma, t) = t\ u(\sigma, t) + \exp(\sigma)/(1-\exp(\sigma))\ u(\sigma, -t)$ , $\sigma \in H$ , $t \in [-1,1]$ .

Let $\sigma_0 \in H$ be defined by $\zeta_0 = \exp(\sigma_0)$ .

A necessary condition for (a) to be solvable is therefore the existence of a solution u of (b) with $u(\sigma_0, t) = f_0(t)$ and $u(\sigma + 2\pi i, t) = u(\sigma, t)$ für all $\sigma \in H$, $t \in [-1,1]$ .

But a solution $u(\sigma, t)$ of (b) defines a map $U : H \to \mathscr{X}$ by $U(\sigma)(t) := u(\sigma, t)$ .

Obviously U is locally bounded and thus weakly complex differentiable (recall that a sequence in $\mathscr{X}$ is weakly convergent if it converges pointwise and the sequence of norms is bounded) , hence holomorphic.

But then Lemma 3.1 tells us $U(\sigma) = V_0(\sigma)(f_0)$ with $V_0$ the appropriate fundamental solution of (+) .

By Proposititon 5.5 the corresponding factor of automorphy has no eigenvalues at all: (a) as stated has no solution, whereas (b) has solutions depending linearly and bijectively on $f_0$.

**Remark:** The naïve attempt to treat the equation (b) numerically by discretizing the real variable t naturally leads to the representations $\Psi$ . This approach seems doubtful as we see that a suitable choice for the abscissas for t ( which then become the eigenvalues of $\Psi(A)$) can cause remarkable changes in the behavior of the resulting finite dimensional problem.

## 8. Hille's Problem

We consider the same algebra $\mathscr{B}$ = BL($\mathscr{X}$) and the operator A ∈ $\mathscr{B}$ as in the previous section.
Let J ∈ $\mathscr{B}$ be defined as Riemann integral by $J(f)(t) := \int_{[0,t]} f(s)\, ds$, $f \in \mathscr{X}$, $t \in [-1,1]$ .

Now J has some elementary but important properties:

**Lemma 8.1:**
*i)* $C_A(J^k) = k\, J^{k+1}$ , $k \geq 1$ .
*ii)* $(C_A)^k(J) = k!\, J^{k+1}$ , $k \geq 0$ .
*iii)* $\rho(J) = 0$ .
*iv)* $\{J\}^c$ is invariant under $C_A$ .
*v)* $\{J\}^c = \{ T \in BL(\mathscr{X}) : T(\tau^k) = k!\, J^k\, T(1)$ for $k \geq 0 \}$ .
*vi)* $J^k(f)(t) = 1/(k-1)! \int_{[0,t]} f(s)\, (t-s)^{k-1}\, ds$, $f \in \mathscr{X}$, $t \in [-1,1]$ , $k \in \mathbb{N}$.
*vii)* $\|J^k\| \leq 1/k!$ , $k \in \mathbb{N}$.

**Proof:**
i+ii+iii): Analogously to Lemma 3.3 we have $A J - J A = J^2$ , that is $\rho(J) = 0$ by using the Theorem of Kleinecke and Shirokov.
By induction we get immediately
$C_A(J^k) = k\, J^{k+1}$ , $k \geq 1$ ,
$(C_A)^k(J) = k!\, J^{k+1}$ , $k \geq 0$ .
iv) Clear from $C_A C_J - C_J C_A = C_{J^2}$ .
v) From $J(\tau^k) = 1/(k+1)\, \tau^{k+1}$ , $k \geq 0$ , it follows for $T \in \{J\}^c$ :
$T(\tau^k) = kJT(\tau^{k-1})$ , $k \geq 1$ , i.e.
$T(\tau^k) = k!\, J^k\, T(1)$ for $k \geq 0$.
The opposite inclusion follows again from Weierstraß' Approximation Theorem, thus
$\{J\}^c = \{ T \in BL(\mathscr{X}) : T(\tau^k) = k!\, J^k\, T(1)$ for $k \geq 0 \}$ .
We see, that a $T \in \{J\}^c$ is uniquely determined by T(1) .
vi) is easily seen by induction using i.
vii) clear from vi . q.e.d.

**Remark:** *8.1 vii is usually used to show that* $\rho(J) = 0$ *and that* 0 *is the only fixed point of* J . *(cf. [Hi], p. 12). Obviously* PSp(J) *is void, so* J *is injective .*
*For fixed* $\lambda \in \mathbb{C}^*$ $(\mathbf{id} + \lambda J)^{-1}$ *is the "solution operator" of a simple linear non-homogenous differential equation in* $\mathscr{X}$*. Solving this differential equation explicitely yields (in line with 8.1i) :*

$( \mathbf{id} + \lambda J )^{-1} = \mathbf{id} - \lambda \exp( - \lambda C_A) (J)$ ,

*so obviously the resolvent* $( \lambda \mathbf{id} - J )^{-1}$ *has an essential singularity in* 0 .
*We note that from 8.1 we get immediately* $\exp(2\pi i C_A)(J) = J(\mathbf{id} - 2\pi i J)^{-1}$.
*Since* $\exp(2\pi i C_A)$ *is an algebra automorphism of* $\{J\}^c$ *we may note especially:*
$\{J\}^c = \{J(\mathbf{id} - 2\pi i J)^{-1}\}^c$.

We fix a $M \in \mathbb{N}$ and consider the differential equation in $\mathscr{B}$ :
(+) $\zeta w'(\zeta) = ( A + J \zeta^M ) w(\zeta) , \zeta \in E^*$ .

If we have $f \in \mathscr{X}$ with $L_f J = J L_f$ , then we get from $J(1) = \tau$ : f is constant,
thus $\{A\}^c \cap \{J\}^c = \mathfrak{J}^c = \mathbb{C}\,\mathbf{id}, \mathfrak{J}^{cc} = \mathscr{B}$ .
Analogously we see that a non-zero eigenvector of $C_A$ corresponding to a non-zero eigenvalue cannot commute with J .

Hille ([Hi], p. 238) considers the case M = 1 and asks whether there exists a Frobenius solution v of (+) on H .
Somewhat disappointed he states that he could not find it.

*For* M > 1 *(+) has a Frobenius solution of type 1 (hence* $q_0 \in \exp(\mathscr{B})$ ) .

**Proof :**
Since Sp(A) = [-1,1], the assertion for M > 2 follows immediately from Proposition 6.8.
Let M = 2 .
Suppose we had indeed a fundamental solution v on H of the form
$v(\sigma) = p^*h(\sigma) \exp(A\sigma )$ with $h \in \mathscr{O}^{\mathscr{B}^*}(E)$ and $h(0) = \mathbf{id}$ .
The differential equation for h (cf. Proposition 6.5) yields the existence of
$x \in \mathscr{B}$ with $( 2\, \mathbf{id} - C_A) (x) = J$ .
x is - if existent - uniquely determined , because 2 is not an eigenvalue of $C_A$ . On the other hand, we have h 2-invariant, so that the existence of x is also sufficient - by Proposition 6.5 - for the existence of h .

Let

$$T_n := \sum_{k=0}^{n} k!/2^{k+1} J^{k+1} \quad , n \geq 0 , \text{ then}$$

$C_A(T_n) = 2\, T_{n+1} - J$ and $C_A^k(x) = 2^k (x - T_{k-1})$ for n, k > 0 .

Obviously $T_n \in \{J\}^c$ and since $\|A\| = 1$, the sequence $( \|T_k\| )$ is bounded.
From the definition of the $T_k$'s it is clear that $T_k$ is quasi-nilpotent for $k \geq 0$.
But Lemma 8.1 viii shows immediately that $(T_k)$ is a Cauchy sequence in $\mathscr{B}$, thus converging uniformly to a $T \in \mathscr{B}$. Hence we may take $x := T$ ; as $T \in \{J\}^c$ we see:
$T(1) = \log ( 2/(2 - \tau) )$ .

q.e.d.

Again we fix an arbitray $M \in \mathbb{N}$ and consider $\mathfrak{G} := \{ T \in BL(\mathscr{B}) : T(\{J\}^c) \subset \{J\}^c \}$.
Recall that because of $C_A C_J - C_J C_A = C_{J^2}$ we have $C_A \in \mathfrak{G}$ .

We may thus consider the differential equation
(++) $\zeta W'(\zeta) = ( C_A + \zeta^M C_J ) W(\zeta)$
in $\mathfrak{G}$ :
If we take a normalized fundamental solution $v_0$ of (+) then ${}^{\alpha}v_0$ is a normalized fundamental solution of (++) , especially ${}^{\alpha}q_0 \in \mathfrak{G}$ . By Lemma 3.9 we may transport (++) via the continuous restriction operator $\mathfrak{G} \to BL(\{J\}^c)$ to $BL(\{J\}^c)$ where we get:
${}^{\alpha}q_0 \mid_{\{J\}^c} = \exp(2\pi i C_A) \mid_{\{J\}^c}$ , resp.
$J(\mathbf{id} - (\sigma-\sigma_0)J)^{-1} = \exp((\sigma-\sigma_0)C_A)(J) = v_0(\sigma) J v_0(\sigma)^{-1}$ ,
$v_0(\sigma)J - Jv_0(\sigma) = (\sigma-\sigma_0) Jv_0(\sigma)J$ ,
especially:
$C_J(q_0) = - 2\pi i\, Jq_0J$ , i.e.. $q_0$ and J do not commute since J is injective .
We have even $\{q_0\}^c \cap \{J\}^c = \mathbb{C}\, \mathbf{id}$ .

If we choose the base point $\sigma_0$ sufficiently to the left in H then Proposition 5.2 yields the existence of $T \in \{J\}^c$ with $q_0 = \exp(2\pi i A) \ \exp(T)$ , with $\|T\|$ small but not zero generically in $\sigma_0$ because otherwise we had $g_0$ constant. More precisely we have :

*There is a non-constant holomorphic function* $G : E \to (\{J\}^c)^*$ *with* $G(0) = \mathbf{id}$ *and*
$g_0(\zeta) = \exp(2\pi i A)\ G(\zeta)$ *for* $\zeta \in E$ .

**Proof:** Proposition 5.2 applied to (+) and resp. (++) in $\mathscr{B}$ resp. $\mathfrak{G}$ yields the existence of a neighborhood U of 0 in E and holomorphic functions $h : U \to \mathcal{B}$ und $H : U \to \mathfrak{G}$ with $h(0) = 0$ , $H(0) = 0$ and $g_0(\zeta) = \exp(2\pi i A)\ \exp(h(\zeta))$ , ${}^{\alpha}g_0(\zeta) = \exp(2\pi i C_A)\ \exp(H(\zeta))$ , $\zeta \in U$ .
Consequently we have $\exp(H(\zeta)) = \exp(C_{h(\zeta)})$ on U; since the exponential function is locally homeomorphic near 0 in $BL(\mathscr{B})$ we have : $H(\zeta) = C_{h(\zeta)}$ on U . From the above we get on the other hand: $\exp(H(\zeta)) \mid_{\{J\}^c} = \mathbf{id}_{\{J\}^c} = \exp (C_{h(\zeta)} \mid_{\{J\}^c})$ .
That means $\Theta(C_{h(\zeta)} \mid_{\{J\}^c})\ C_{h(\zeta)} \mid_{\{J\}^c} = 0$ . Shrinking U eventually further $\Theta(C_{h(\zeta)})$ becomes regular in $BL(\mathscr{B})$ and we get $h(\zeta) \in \{J\}^c$ and thus $\exp(h(\zeta)) \in \{J\}^c$.

Since $G(\zeta) := \exp(-2\pi iA)g_0(\zeta)$ is holomorphic on $E$ with $C_J(G(\zeta)) = 0$ on an open set containing 0 we get $G(\zeta) \in \{J\}^c$ for all $\zeta \in E$.

From example 5.7 we deduce that G is not constant.

q.e.d.

For $\zeta \in E$ we have:

$C_{f(\zeta)}(g_0(\zeta)) = \zeta g_0'(\zeta) = \zeta \exp(2\pi iA)\, G'(\zeta)$,

$\zeta G'(\zeta) = \exp(-2\pi iA)\, C_{f(\zeta)}(\exp(2\pi iA)\, G(\zeta)) = C_A(G(\zeta)) + \zeta^M \exp(-2\pi iA)\, C_J(\exp(2\pi iA))\, G(\zeta)$

i.e.

$\zeta G'(\zeta) = C_A(G(\zeta)) - 2\pi i\zeta^M J^2(\mathbf{id} + 2\pi iJ)^{-1}G(\zeta)$ .

Obviously G cannot commute generically with A .

For $C := G'(0)$ we get from this differential equation:

$C = C_A(C) - 2\pi i\, J^2(\mathbf{id} + 2\pi iJ)^{-1} = C_A(C + \log(\mathbf{id} + 2\pi iJ))$, if $M = 1$ ,

resp. $C = 0$ für $M > 1$ .

C belongs in each case to the image of $C_A$ .

Using the continuous pseudo-involution $^\sim$ on $\mathscr{B}$ we get (c.f. sections 5, 7):

$F(\zeta) := (G^x(\zeta))^{-1} = \exp(2\pi i\, C_A)(G(\zeta))$ for $\zeta \in E$ .

For $D := F'(0)$ in the case $M = 1$ we obtain:

$D - C_A(D) = \exp(2\pi i\, C_A)\,(-2\pi i\, J^2(\mathbf{id} + 2\pi iJ)^{-1}) = -2\pi i\, J^2(\mathbf{id} - 2\pi iJ)^{-1}$ .

We are now ready to prove:

*If* $M = 1$ *then* (+) *has no Frobenius solution of any type .*

**Proof:**

Suppose we had a Frobenius solution v of type n of (+), i.e. there are $n \in \mathbb{N}$ and $h_0, \ldots, h_{n-1} \in \mathscr{O}^{\mathscr{B}}(E)$ with $h_0(0) = (n-1)!\mathbf{e}$ , $h_k(0) = 0$, $k = 1,\ldots, n-1$ , $h_{n-1}$ not identically zero and $v(\sigma) = (h_0(e^\sigma) + \sigma h_1(e^\sigma) + \ldots + \sigma^{n-1}h_{n-1}(e^\sigma)) \exp(A\sigma)$ , $\sigma \in H$ , defines a fundamental solution of (+) on H.

By Proposition 2.2 each $h_k$, $k = 1,\ldots, n-1$ , has in 0 a zero of order 1. The leading coefficient $c_{n-1}$ of $h_{n-1}$ is an eigenvector of $C_A$ corresponding to the eigenvalue 1 (if $n > 1$ ) resp. $c_{n-1} = \mathbf{id}$ for $n = 1$ .

For $u(\sigma) := (h_0(e^\sigma) + \sigma h_1(e^\sigma) + \ldots + \sigma^{n-1}h_{n-1}(e^\sigma)$ , $\sigma \in H$, we have

$\exp(2\pi iA)\; p^*G = (\mathfrak{g}_0^{-1})^*u\; \exp(2\pi iA)\; u^{-1}$

$(\mathfrak{g}_0^{-1})^*u = \exp(2\pi i\, C_A)\,(p^*G\, u) = p^*F\; \exp(2\pi i\, C_A)(u)$ , hence

$h_{n-1} = F \exp(2\pi i\, C_A)(h_{n-1})$ .It follows

$h_{n-1}' = F' \exp(2\pi i\, C_A)(h_{n-1}) + F \exp(2\pi i\, C_A)(h_{n-1}')$ .

Suppose we had n = 1 . Then for $x := h_0'(0)$ we get:

$x = C_A(x) + J$ ,

$x = D + \exp(2\pi i\, C_A)(x)$ .

With

$$S_n := \sum_{k=0}^{n} k!\, J^{k+1} \quad , n \geq 0 , \quad \text{we have}$$

$C_A(S_n) = S_{n+1} - J$ and hence

$C_A{}^k(x) = x - S_{k-1}$ for $k > 0$.

We thus get for appropriate complex $\lambda$ :

$$(\lambda\, \mathbf{id} - C_A)^{-1}(x) = x/(\lambda - 1) - \sum_{k=1}^{\infty} S_{k-1}/\lambda^k .$$

That means that x commutes with J .

But then necessarily $x(1) = = -\log(1 - \tau)$ which obviously is not continuous on the whole interval [-1,1]. Contradiction.

Suppose then n > 1.

$C_A(x) = C_A(D) + \exp(2\pi i\, C_A)(C_A(x))$ ,

$x - J = D + 2\pi i\, J^2(\mathbf{id} - 2\pi i J)^{-1} + \exp(2\pi i\, C_A)(x) - \exp(2\pi i\, C_A)(J)$

$D + \exp(2\pi i\, C_A)(x) - J = D + 2\pi i\, J^2(\mathbf{id} - 2\pi i J)^{-1} + \exp(2\pi i\, C_A)(x) - \exp(2\pi i\, C_A)(J)$

$- J = 2\pi i\, J^2(\mathbf{id} - 2\pi i J)^{-1} - \exp(2\pi i\, C_A)(J) = 2\pi i\, J^2(\mathbf{id} - 2\pi i J)^{-1} - J(\mathbf{id} - 2\pi i J)^{-1}$

$- J = J(\mathbf{id} - 2\pi i J)^{-1}(2\pi i\, J - \mathbf{id})$ .

and Proposition 2.2. yields with $\zeta h_{n-1}'(\zeta) = f(\zeta)h_{n-1}(\zeta) - h_{n-1}(\zeta)\, A$ , $f(\zeta) = A + \zeta\, J$ :

$f\, h_{n-1} - h_{n-1} A = \zeta\ F' \exp(2\pi i\, C_A)(h_{n-1}) + F \exp(2\pi i\, C_A)(f\, h_{n-1} - h_{n-1} A)$

$= \zeta\ F'\ F^{-1} h_{n-1} + F \exp(2\pi i\, C_A)(f\, h_{n-1}) - h_{n-1} A$

$f\, h_{n-1} = \zeta\ F'\ F^{-1} h_{n-1} + F \exp(2\pi i\, C_A)(f\, h_{n-1})$

??????????

q.e.d.

## 9. The Hilbert Space Case

Let $\mathfrak{H} := L^2([0,1])$ be the Hilbert space of (equivalence classes of) of complex-valued functions on $I:=[0,1]$, which are square-integrable in the sense of Lebesgue.
Then we can consider $\mathscr{X} = C[0,1]$ as subspace of $\mathfrak{H}$; but pointwise multiplication with continuous functions does not affect square-integrability so that for $f \in \mathscr{X}$ we can consider $L_f$ as element of $BL(\mathfrak{H})$.
Especially we have:

*If a sequence* $(f_n)$ *in* $\mathfrak{H}$ *converges to zero (with respect to the* $L^2$*-norm) then this holds equally for the sequence* $(pf_n)$ *for any* $p \in \mathscr{X}$.

**Proof:** $(\|pf_n\|_2)^2 = \int_I |p|^2 \, |f_n|^2 \, dt \leq (\|p\|_\infty \, \|f_n\|_2)^2$ . q.e.d.

The map $\iota: \mathscr{X} \to BL(\mathfrak{H})$ , $\iota(f)(g) := fg$ , $f \in \mathscr{X}$, $g \in \mathfrak{H}$ , obviously defines an injective *-homomorphism between B*-algebras, hence an isometry. We can thus identify $\mathscr{X}$ with its image $\iota(\mathscr{X})$.
The operators $A_0$, $N_0$ as defined in section 7 may thus be considered as elements of $\mathscr{B} := BL(\mathfrak{H})$. As above we dentote by $\mathscr{P}$ the $\mathbb{C}$-algebra of polynomial functions on [0,1], $\mathscr{P} \subset \mathfrak{H}$.
Since $\mathfrak{H}$ admits of a countable othonormal basis consisting of polynomials, $\mathscr{P} \subset \mathfrak{H}$ is dense with respect to the $L^2$-norm.
Let $\mathscr{C} := \{ T \in \mathscr{B} : T|_{\mathscr{P}} = L_{T(1)} \}$; $\mathscr{C}$ is clearly a linear subspace of $\mathscr{B}$. A $T \in \mathscr{C}$ is uniquely determined by $T(1) \in \mathfrak{H}$.
If $(T_n)$ is a sequence in $\mathscr{C}$ converging to $T \in \mathscr{B}$ with respect to the operator norm, then we have for $p \in \mathscr{P}$:
$T(p) - T(1)p = ( T(p) - T_n(p) ) + (T_n(p) - T(1)p) = ( T(p) - T_n(p) ) + (T_n(1)-T(1))\, p.$
The right hand side tends to 0 for $n \to \infty$, i.e. $\mathscr{C}$ is closed in $\mathscr{B}$.
Like in section 7 we see immediately $\{A_0\}^c \subset \mathscr{C}$. On the other hand, if $T \in \mathscr{C}$, $p \in \mathscr{P}$, then $A_0T(p) = \tau p T(1) = TA_0(p)$ , $T \in \{A_0\}^c$ ; thus $\{A_0\}^c = \mathscr{C}$. Since $A_0$ is hermitian ( i.e. selfadjoint with respect to usual algebra involution * on $\mathscr{B}$ ) , $\{A_0\}^c$ is a C*-Algebra.
$N_0$ is hermitian too, thus $\{N_0\}^c$ is also a C*-algebra.
Especially, $\{A_0\}^c$ and $\{N_0\}^c$ are semi-simple.

For $T \in \mathscr{B}$ let $T^*$ denote the adjoint operator in the Hilbert space sense and $T^\sim \in \mathscr{B}$ be defined by $T^\sim(f) = (T(f^*))^*$ with pointwise conjugation in $\mathfrak{H}$ (see section 3.5). Obviously $\mathscr{C}$ is closed under $^\sim$ .

By definition of the scalar product in $\mathfrak{H}$ ( $<f,g> := \int_I fg^*dt$ ) we get $T^{*\sim} = T^{\sim *}$ for $T \in \mathcal{B}$.

**Proof:** For $p,q \in \mathcal{P}$ we have $<T^{*\sim}(q), p> = < (T^*(q^*) )^* , p> = < T^*(q^*),p^*>^* = <q^*, T(p^*)>^* =$ $<q,T^{\sim}(p)> = < T^{\sim *}(q),p>$. q.e.d.

But for $T \in \mathcal{C}$ we have $T^* = T^{\sim}$ , i.e. for $S,T \in \mathcal{C}$: $ST = (T^*S^*)^* = (T^{\sim}S^{\sim})^{\sim} = TS$, $\{A_0\}^c$ is commutative. So if $T \in \{A_0\}^c$ is hermitian, then $T = T^{\sim}$ , thus $T(1) = T(1)^*$ .
Let $T \in \{A_0\}^c$ be hermitian and $f := T(1) \in \mathfrak{H}$ , then $f \in L^1([0,1])$ ; but from $T(1) = T(1)^*$ we get $T^k(1) = (T(1))^k$ for natural k, so $f^k \in L^1([0,1])$ , hence $f \in L^p([0,1])$ for all $p > 0$ . That means finally: $\{A_0\}^c = \{ L_f : f \in L^\infty([0,1]) \}$.
It would be interesting to describe $X(\{A_0\}^c)$ .

If $\chi$ denotes the characteristic function of a measurable subset of I with positive measure, then $\chi \in \mathfrak{H}$ and $P_\chi := L_\chi$ defines an idempotent $\in \mathcal{C}$ (indeed an orthogonal projection in the Hilbert space sense).
Thus the inclusion $V(A_0) = \iota(\mathcal{X}) \subset \{A_0\}^c = \iota(\mathcal{X})^c$ is strict.
Since $A_0$ is hermitian, $Sp_{\mathcal{B}}(A_0)$ is real and therefore $Sp_{\mathcal{B}}(A_0) = Sp_{\iota(\mathcal{X})}(A_0) = I$ , hence $\rho(A_0) = 1$.
$PSp(A_0)$ ist obviously void.
With the isometry $\iota: \mathcal{X} \rightarrow BL(\mathfrak{H})$ one deduces analogously to section 7: $Sp({}^{C}A_0) = [-1,1]$ .

We observe, that a nilpotent $T \in \mathcal{B}$ cannot be normal, for otherwise from $TT^* = T^*T$ we get $T^*T$ nilpotent too, that is - since $T^*T$ is hermitian - : $0 = \rho(T^*T) = \|T^*T\| = \|T\|^2$ , $T = 0$.

*We have* $PSp({}^{C}A_0) \subset (-1,1)$ .

**Proof:** As in section 7 it is sufficient to prove that 1 is not an eigenvalue of ${}^{C}A_0$ .
Assume the contrary, then we have a $T \in \mathcal{B}$ with $T \neq 0$ and $T(p) = E_1 (p)T(1)$ for all $p \in \mathcal{P}$.
Consider $p := 2\tau - 1 \in \mathcal{P}$ with $E_1(p) = 2\tau - 3$ and let $h := T(1) \in \mathfrak{H}$ .
For any natural k we get :
$h = ( A_0 - 3/2\, \mathbf{id})^{-k} T(p^k) ) / 2^k$
$\| h\|_2 \leq \| T\| \, \| p^k\|_2$ .
But one easily checks $\| p^k\|_2 = 1/\sqrt{(2k+1)}$ , so $h = 0$ , $T = 0$. Contradiction.

q.e.d.

Anlogously to section 7 we consider now the differential equation :

(+) $\zeta w'(\zeta) = ( A_0 + N_0 \zeta/(1-\zeta) ) w(\zeta) , \zeta \in E^* = \{ \zeta \in \mathbb{C} : 0 < |\zeta| < 1\}$ .

Again we have $\mathfrak{J}^c = \{A_0\}^c \cap \{N_0\}^c$ and with $A' := A_0 - \mathbf{id}/2$:

$\{A_0\}^c \cap \{N_0\}^c = \{A'^2\}^{cc} = \mathfrak{J}^c$ , $\mathfrak{J}^{cc} = \{A'^2\}^c$ , $PSp(C_{A0} \mid \{A'^2\}^c) = \{0\}$ .

If now $Q_0$ is the factor of automorphy of a normalized (at real $\sigma_0 \in H$ ) fundamental solution $V$ of (+) on H, then by Proposition 5.6 we have :

$\{A'^2\}^{cc} = \mathfrak{J}^c \subset \{A_0\}^c \cap \{Q_0\}^c \subset \{Q_0\}^c \subset \{A'^2\}^c$.

By propositions 5.1 and 5.11 $\{Q_0\}^c$ is commutative, semi-simple and symmetric with respect to $^\sim$ , $\{Q_0\}^c = \mathscr{A}$ and $\Gamma_0 : \mathscr{A} \rightarrow \{A_0\}^c$ is an injective *-homomorphism with respect to $^\sim$ .

Clearly $\mathscr{A}$ contains non-trivial idempotents and $Q_0 = \cos 2\pi A_0 + iv$ , with $v \in \{A'^2\}^c$ , $v^2 = (\sin 2\pi A_0)^2$ , $v = v^\sim$ , because $\Gamma_0$ is injective. Again $\Gamma_0 : \mathscr{A} \rightarrow \Gamma_0(\mathscr{A})^-$ is equivalent to the Gelfand representation of $\mathscr{A}$.

Observe that $\cos 2\pi A_0 \in \{A'^2\}^{cc}$ .

Obviously $Q_0$ is normal with respect to * if and only if $vv^* = v^*v$ . Consequently if $Q_0$ is normal with respect to *, then $\{Q_0\}^c = \{Q_0{}^*\}^c$ , so that $^\sim$ and * must coincide on $\{Q_0\}^c$ because it is symmetric with respect to $^\sim$ and semi-simple. But then $Q_0$ is unitary with respect to *, $v = v^*$. So the set of real $\sigma_0 \in H$ for which the corresponding $Q_0$ is normal cannot have a cluster point in H, because otherwise we had $g_0{}^x(\zeta) = g_0{}^+(\zeta)$, where for $w \in \mathscr{O}^{\mathscr{B}}(E)$ $w^+ \in \mathscr{O}^{\mathscr{B}}(E)$ is defined by $w^+(\zeta) = w(\zeta^*)^*$ , $\zeta \in E$ . But this would imply $g_0$ to be constant in contradiction to Proposition 5.6.

With this argument we see also that the set of points $\zeta \in E$ where $g_0(\zeta)$ commutes with $A_0$ has no cluster point in E.

Let us show:

*The differential equation (+) has no fundamental solution of the form:*

$v(\sigma) = p^*h(\sigma) \exp(A_0\sigma)$ *with* $h \in \mathscr{O}^{\mathscr{B}*}(E)$ *and* $h(0) = \mathbf{id}$ .

**Proof:**

If we had such a fundamental solution then $Q_0$ would have a logarithm $2\pi i y_0$ in $\{Q_0\}^c \subset \{A'^2\}^c$ . But then we had $V$ of the form $V(\sigma) = p^*m(\sigma) \exp(y_0(\sigma - \sigma_0))$. The values of m commute with $A'^2$ and Proposition 6.4 shows that m is holomorphic and regular in 0 . But then again with 6.4 we may assume that the values of h belong to $\{A'^2\}^c$ .

As in section 7 we get for $T := h'(0) \in \{A'^2\}^c$: $C_{A0}(T) = T - N_0$ .

By Lemma 3.8 and the structure of $\{A_0\}^c = \{A'\}^c$ we get a non-zero $f \in L^\infty([0,1])$ with $C_{A0}(T) = L_f N_0$ . That means $T = L_{1+f} N_0$ , $L_f N_0 = L_{(2\tau-1)(1+f)} N_0$ , $f = (2\tau-1)(1+f)$ , $f = (2\tau-1)/(2(1-\tau))$. But f is obviously essentially unbounded. q.e.d.

By this result and Proposition 2.2 we can conclude that (+) has no Frobenius solution of any type.

Proposition 6.10 tells us $A_0 \notin \Gamma_0(\mathscr{A})$, $\Gamma_0(\mathscr{A})$ is a full selfadjoint subalgebra of $\{A_0\}^c$ which contains $\exp(2\pi i A_0)$ but not $A_0$. $Q_0$ and $\exp(2\pi i A_0)$ are not conjugate in $\mathscr{B}^*$. Moreover, $A_0$ does not belong to the uniform closure of $\Gamma_0(\mathscr{A})$.
As in 6.13 we see $\Gamma_0(\mathscr{A}) \neq \Gamma_0(\mathscr{A})^{cc} = \{A_0\}^c$ with respect to $\mathscr{B}$, i.e. $\Gamma_0(\mathscr{A})$ is not weakly closed in $\mathscr{B}$ (cf. [Ne]).

The abundance of idempotents in $\{A_0\}^c$ prevents us from continuing the argumentation as in section 7.
Suppose $Q_0$ had a logarithm $2\pi i y_0$ in $\{Q_0\}^c \subset \{A'^2\}^c$. Since $\{Q_0\}^c$ is commutative, symmetric with respect to ~ and semi-simple, we have $y_0 = y_0^{\sim}$ and $\{Q_0\}^c = \{y_0\}^c$, so we get
$\cos 2\pi y_0 = \cos 2\pi A_0$ and $v = \sin 2\pi y_0$.
Since $Q_0$ is generically not normal we may suppose that $y_0$ is not normal either and that consequently v and $Q_0$ do not commute with $A_0$.
According to Proposition 6.1 there is an integer linear combination of idempotents $p_i$ from $\{A_0\}^c$, say $r = n_1 p_1 + .... + n_k p_k$, with $\Gamma_0(y_0) = A_0 + r$. We may suppose that all $n_i > 1$ and that they all are distinct.
With $V(\sigma) = p^* m(\sigma) \exp(y_0(\sigma - \sigma_0)$ and $h := h_{y_0}$ we see that m is a normalized fundamental solution of
(++) $\zeta m'(\zeta) = ( f(\zeta) - h(\zeta))\, m(\zeta)$.
Using our usual notation we get for this equation :
$a_0 = -r$, $q_0 = \exp(2\pi i a_0) = \mathbf{id}$,
so we find for the corresponding objects $\mathscr{A}$ and $\Gamma_0$, which we mark by a prime:
$\{Q_0\}^c \subset \mathscr{A}'$, $\Gamma_0'(z) = \Gamma_0(z)$ for all $z \in \{Q_0\}^c$.
By Proposition 6.2 we get a $h \in \mathscr{O}^{\mathscr{B}^*}(E)$ with $h(0) = \mathbf{e}$ and $z \in \mathscr{B}^*_0$ with
$p^* m(\sigma) = p^* h(\sigma) \exp(- r\, \sigma)\, z$; set $y := z\, y_0\, z^{-1}$.

## 10. Open Questions

1.

Is there an unital complex Banach algebra $\mathscr{B}$, in which exp($\mathscr{B}$) lies strictly but open in $\mathscr{B}^*_0$ ?

2.

Do there exist any relations between the properties of a Fuchsian singularity of the differential equation (*)

a) to admit of a Frobenius solution

b) to be a *Stelle der Bestimmtheit* ?

3.

If (*) admits of a Frobenius solution then there is clearly one with minimal type. To what extent is the type determined by (*) , is there a maximal type ?

4.

Consider the second example of section 7 at the singular point 0 and in the full algebra $\mathscr{B}$ = BL($\mathscr{X}$) . Does $Q_0$ admit of a logarithm in $\mathscr{B}$ ?

5.

Give an answer to Hille's original question in section 8.

6.

The results at the end of section 6 for the matrix case still seem to be rather incomplete. Can they be ameliorated, perhaps with the ideas of Anosov and Bolibruch [AB] ?

7.

Describe the Gelfand space X( $L^\infty$([0,1]) ) .

8.

Prove that exp(BL($\mathfrak{H}$) ) is not open with the methods of this paper (section 9) .

9.

Prove or disprove the conjecture that if for fixed $\mathscr{B}$ all factors of automorphy of any fuchsian differential equation (*) belong to exp($\mathscr{B}$) then exp($\mathscr{B}$) is open .

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